Eureka Math™ Homework Helper

2015–2016

Algebra II
Module 1
Lessons 1–40

Eureka Math, A Story of Functions®

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Lesson 1: Successive Differences in Polynomials

Investigate Successive Differences in Sequences

Create a table to find the third differences for the polynomial $45x - 10x^2 + 3x^3$ for integer values of x from 0 to 4.

I calculate first differences by subtracting the outputs for successive inputs. The second differences are the differences of successive first differences, and the third differences are differences between successive second differences.

x	y	First Differences	Second Differences	Third Differences
0	$\mathbf{0}$			
		$\mathbf{38 - 0 = 38}$		
1	$\mathbf{38}$		$\mathbf{36 - 38 = -2}$	
		$\mathbf{74 - 38 = 36}$		$\mathbf{16 - (-2) = 18}$
2	$\mathbf{74}$		$\mathbf{52 - 36 = 16}$	
		$\mathbf{126 - 74 = 52}$		$\mathbf{34 - 16 = 18}$
3	$\mathbf{126}$		$\mathbf{86 - 52 = 34}$	
		$\mathbf{212 - 126 = 86}$		
4	$\mathbf{212}$			

I notice the third differences are constant, which should be true for a polynomial of degree 3.

Write Explicit Polynomial Expressions for Sequences by Investigating Their Successive Differences

Show that the set of ordered pairs (x, y) in the table below satisfies a quadratic relationship. Find the equation of the form $y = ax^2 + bx + c$ that all of the ordered pairs satisfy.

x	0	1	2	3	4
y	6	9	22	45	78

I know that if the second differences are constant, and the first differences are not, then the relationship is quadratic.

x	y	First Differences	Second Differences	Third Differences
0	6			
		3		
1	9		**10**	
		13		
2	22		**10**	
		23		
3	45		**10**	
		33		
4	78			

$\boldsymbol{y = ax^2 + bx + c}$
$\boldsymbol{a = 5}$ ***and*** $\boldsymbol{c = 6}$

To model the ordered pairs with a quadratic equation, I need to find values of a, b, and c in the equation $y = ax^2 + bx + c$.

$\boldsymbol{9 = a(1)^2 + b(1) + c}$
$\boldsymbol{9 = 5(1)^2 + b + 6}$
$\boldsymbol{9 = 11 + b}$
$\boldsymbol{b = -2}$
$\boldsymbol{y = 5x^2 - 2x + 6}$

I found the second differences to be 10, and I know that the second differences are equal to $2a$, so it must be that $a = 5$. I know the y-intercept is equal to c and $5(0)^2 + b(0) + c = 6$, so $c = 6$.

I solved $y = 5x^2 + bx + 6$ when $x = 1$ and $y = 9$ to find the value of b.

Lesson 2: The Multiplication of Polynomials

Use a Table to Multiply Two Polynomials

1. Use the tabular method to multiply $(5x^2 + x + 3)(2x^3 - 5x^2 + 4x + 1)$, and combine like terms.

I multiply the terms in the corresponding row and column to fill in each cell in the table.

The top row contains the terms of the first polynomial.

The right-hand column contains the terms of the second polynomial.

$5x^2$	x	3	
$10x^5$	$2x^4$	$6x^3$	$2x^3$
$-25x^4$	$-5x^3$	$-15x^2$	$-5x^2$
$20x^3$	$4x^2$	$12x$	$4x$
$5x^2$	x	3	1

For example, $x \cdot 2x^3 = 2x^4$.

Also, $3 \cdot (-5x^2) = -15x^2$.

$10x^5$

$-23x^4$

$21x^3$

$-6x^2$ $13x$ 3

I add the like terms found along the upper right to lower left diagonals in the table. For example, $6x^3 - 5x^3 + 20x^3 = 21x^3$.

$$(5x^2 + x + 3)(2x^3 - 5x^2 + 4x + 1) = 10x^5 - 23x^4 + 21x^3 - 6x^2 + 13x + 3$$

Now that I have combined like terms, I write the product as the sum of the terms from highest to lowest degree.

Use the Distributive Property to Multiply Two Polynomials

2. Use the distributive property to multiply $(6x^2 + x - 2)(7x + 5)$, and then combine like terms.

I distribute each term in the first polynomial to the second polynomial. Then I multiply each term of the first polynomial by each term in the second polynomial.

$$\begin{aligned}(6x^2 + x - 2)(7x + 5) &= 6x^2(7x + 5) + x(7x + 5) - 2(7x + 5)\\ &= 6x^2(7x) + 6x^2(5) + x(7x) + x(5) - 2(7x) - 2(5)\\ &= 42x^3 + 30x^2 + 7x^2 + 5x - 14x - 10\\ &= 42x^3 + 37x^2 - 9x - 10\end{aligned}$$

Apply Identities to Multiply Polynomials

Multiply the polynomials in each row of the table.

If I recognize an identity, I can use the formula associated with it to multiply polynomials. If I do not recognize an identity, I can multiply polynomials using the tabular method or distributive property.

I recognize this product as a difference of squares: $(a + b)(a - b) = a^2 - b^2$. In this case, a is x and b is $2y^2$.

3. $(x + 2y^2)(x - 2y^2)$ $\quad (x)^2 - (2y^2)^2 = x^2 - 4y^4$

4. $(3x - 7y)^2$ $\quad (3x)^2 + 2(3x)(-7y) + (-7y)^2 = 9x^2 - 42xy + 49y^2$

I know that $(a - b)^2 = a^2 - 2ab + b^2$. In this case, a is $3x$ and b is $-7y$.

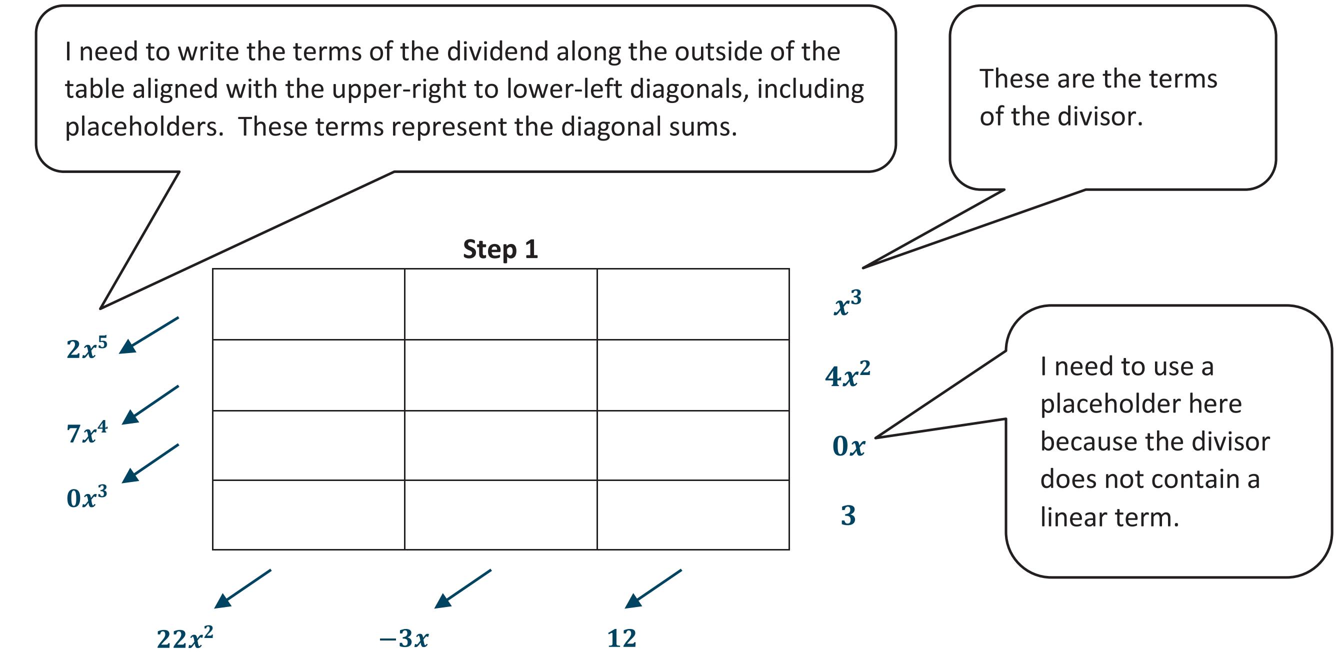
5. $(x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ $\quad x^{6+1} - 1 = x^7 - 1$

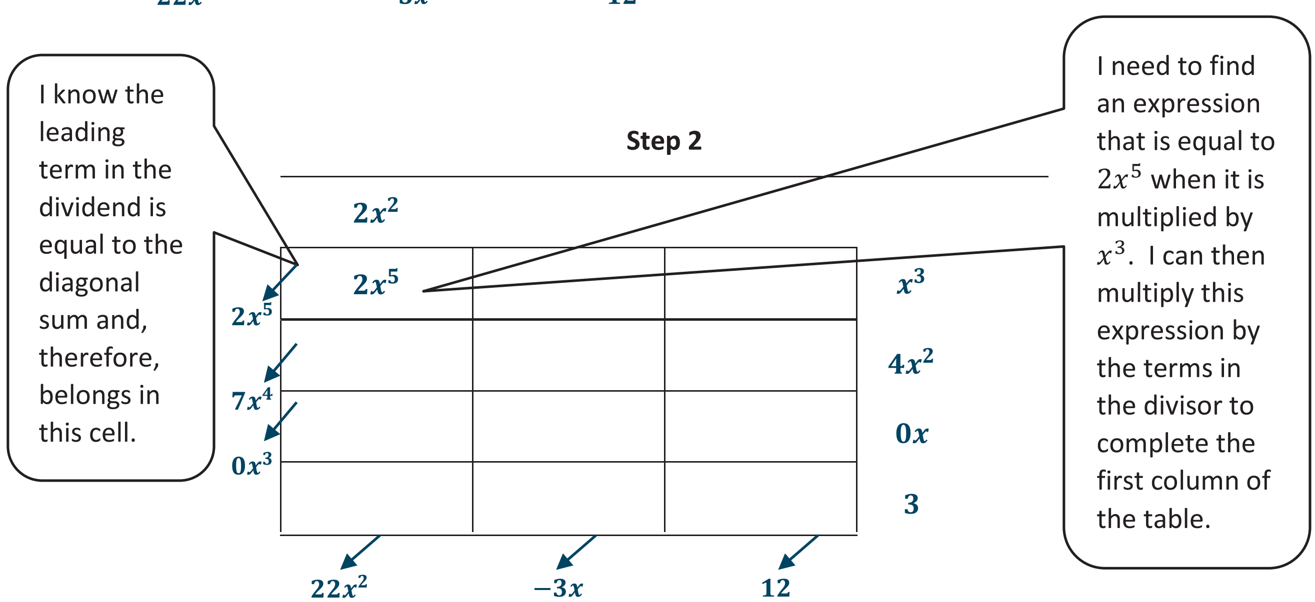
I recognize this pattern:
$$(x - 1)(x^n + x^{n-1} + x^{n-2} + \cdots + x + 1) = x^{n+1} - 1$$

Lesson 3: The Division of Polynomials

Use a Table to Divide Two Polynomials

Use the reverse tabular method to find the quotient: $(2x^5 + 7x^4 + 22x^2 - 3x + 12) \div (x^3 + 4x^2 + 3)$.





I know this value, added to $8x^4$, must equal the diagonal sum, $7x^4$.

I need to find an expression that equals $-1x^4$ when multiplied by x^3. I can multiply this expression by the remaining terms in the divisor to complete the second column of the table.

Step 3

	$2x^2$	$-x$		
	$2x^5$	$-1x^4$		x^3
$2x^5$	$8x^4$			$4x^2$
$7x^4$	$0x^3$			$0x$
$0x^3$	$6x^2$			3
	$22x^2$	$-3x$	12	

I know this value, added to $-4x^3 + 0x^3$, must equal the diagonal sum, $0x^3$.

I need to find an expression that equals $4x^3$ when multiplied by x^3. I can multiply this expression by the remaining terms in the divisor to complete the third column of the table and confirm the constant term.

Step 4

	$2x^2$	$-x$	$+4$	
	$2x^5$	$-1x^4$	$4x^3$	x^3
$2x^5$	$8x^4$	$-4x^3$		$4x^2$
$7x^4$	$0x^3$	$0x^2$		$0x$
$0x^3$	$6x^2$	$-3x$		3
	$22x^2$	$-3x$	12	

The expression across the top row of the table is the quotient.

$$(2x^5 + 7x^4 + 22x^2 - 3x + 12) \div (x^3 + 4x^2 + 3) = 2x^2 - x + 4$$

Lesson 4: Comparing Methods—Long Division, Again?

1. Is $x+5$ a factor of x^3-125? Justify your answer using the long division algorithm.

$$\begin{array}{r|llll}
\cline{2-5}
x+5 & x^3 & +0x^2 & +0x & -125
\end{array}$$

I know that, just like with long division of integers, the divisor is a factor of the dividend if the remainder is zero.

$$\begin{array}{r|llll}
 & & x^2 & & \\
\cline{2-5}
x+5 & x^3 & +0x^2 & +0x & -125 \\
 & -(x^3 & +5x^2) & & \\
\cline{2-3}
 & & -5x^2 & +0x &
\end{array}$$

$$\begin{array}{r|llll}
 & & x^2 & -5x & \\
\cline{2-5}
x+5 & x^3 & +0x^2 & +0x & -125 \\
 & -(x^3 & +5x^2) & & \\
\cline{2-3}
 & & -5x^2 & +0x & \\
 & & -(-5x^2 & -25x) & \\
\cline{3-4}
 & & & 25x & -125
\end{array}$$

I need to find an expression that multiplied by x will result in x^3, the first term in the dividend. Then I multiply this expression by the terms of the divisor, subtract them from the dividend, and bring down the next term from the dividend.

$$\begin{array}{r|llll}
 & & x^2 & -5x & +25 \\
\cline{2-5}
x+5 & x^3 & +0x^2 & +0x & -125 \\
 & -(x^3 & +5x^2) & & \\
\cline{2-3}
 & & -5x^2 & +0x & \\
 & & -(-5x^2 & -25x) & \\
\cline{3-4}
 & & & 25x & -125 \\
 & & & -(25x & +125) \\
\cline{4-5}
 & & & & -250
\end{array}$$

I repeat the steps in the division algorithm until the remainder has a degree less than that of the divisor.

Because long division does not result in a zero remainder, I know that $x+5$ is not a factor of x^3-125.

Lesson 5: Putting It All Together

Operations with Polynomials

For Problems 1–2, quickly determine the first and last terms of each polynomial if it was rewritten in standard form. Then rewrite each expression as a polynomial in standard form.

1. $\frac{x^3-8}{x-2}+\frac{x^3-x^2-10x-8}{x-4}$

For each quotient, I can find the term with the highest degree by dividing the first term in the numerator by the first term in the denominator. Since the highest-degree terms are like terms, I combine them to find the first term of the polynomial.

I can use the same process with the last terms of the quotients to find the last term of the polynomial.

The first term is

$\boldsymbol{\frac{x^3}{x}+\frac{x^3}{x}=x^2+x^2=2x^2}$,

and the last term is $\boldsymbol{\frac{-8}{-2}+\frac{-8}{-4}=4+2=6}$.

To rewrite the polynomial, I can use the reverse tabular method or division algorithm (from Lessons 3–4) to find each quotient and then add the resulting polynomials.

$$
\begin{array}{r|rrrr}
 & x^2 & +2x & +4 & \\
\hline
x-2 & x^3 & +0x^2 & +0x & -8 \\
 & -(x^3 & -2x^2) & & \\
\hline
 & & 2x^2 & +0x & \\
 & & -(2x^2 & -4x) & \\
\hline
 & & & 4x & -8 \\
 & & & -(4x & -8) \\
\hline
 & & & & 0
\end{array}
\qquad
\begin{array}{r|rrrr}
 & x^2 & +3x & +2 & \\
\hline
x-4 & x^3 & -x^2 & -10x & -8 \\
 & -(x^3 & -4x^2) & & \\
\hline
 & & 3x^2 & -10x & \\
 & & -(3x^2 & -12x) & \\
\hline
 & & & 2x & -8 \\
 & & & -(2x & -8) \\
\hline
 & & & & 0
\end{array}
$$

Standard form: $\boldsymbol{\frac{x^3-8}{x-2}+\frac{x^3-x^2-10x-8}{x-4}=\left(x^2+2x+4\right)+\left(x^2+3x+2\right)=2x^2+5x+6}$

2. $(x-5)^2+(2x-1)(2x+1)$

I can multiply the first terms in each product and add them to find the first term of the polynomial. I can multiply the constant terms in each product and add them to find the last term.

The first term is $x^2+4x^2=5x^2$; the last term is $25-1=24$.

$$\left(x^2+2(x)(-5)+(-5)^2\right)+\left((2x)^2-(1)^2\right)=\left(x^2-10x+25\right)+\left(4x^2-1\right)=5x^2-10x+24$$

I can use the identities $(a-b)^2=a^2-2ab+b^2$ and $(a-b)(a+b)=a^2-b^2$ to multiply the polynomials.

Lesson 6: Dividing by $x - a$ and by $x + a$

Compute each quotient.

I can factor the numerator using the difference of squares identity:
$x^2 - a^2 = (x - a)(x + a)$

1. $\frac{9x^2-25}{3x+5}$

$$\frac{(3x)^2 - 5^2}{3x + 5} = \frac{(3x - 5)(3x + 5)}{3x + 5} = 3x - 5$$

2. $\frac{64x^3-27}{4x-3}$

$$\frac{(4x)^3 - 3^3}{4x - 3} = \frac{(4x - 3)\left((4x)^2 + 3(4x) + 3^2\right)}{4x - 3}$$
$$= 16x^2 + 12x + 9$$

I can rewrite the numerator as $(4x)^3 - 3^3$. Then I can factor this expression using the difference of cubes identity:
$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$.

3. $\frac{8x^3+1}{1+2x}$

$$\frac{(2x)^3 + 1^3}{1 + 2x} = \frac{(2x + 1)\left((2x)^2 - 1(2x) + 1^2\right)}{2x + 1} = 4x^2 - 2x + 1$$

I recognize that I can use an identity to factor the numerator for Problems 4 and 5:
$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x^1 + a^{n-1})$

4. $\frac{x^9-1}{x-1}$

$$\frac{(x - 1)\left(x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\right)}{x - 1}$$
$$= x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

5. $\frac{x^5-32}{x-2}$

$$\frac{x^5 - 2^5}{x - 2} = \frac{(x - 2)\left(x^4 + 2x^3 + 2^2x^2 + 2^3x + 2^4\right)}{x - 2}$$
$$= x^4 + 2x^3 + 4x^2 + 8x + 16$$

Lesson 7: Mental Math

Use Polynomial Identities to Perform Arithmetic

1. Using an appropriate polynomial identity, quickly compute the product $52 \cdot 28$. Show each step. Be sure to state your values for x and a.

$$\boldsymbol{x = \frac{52 + 28}{2} = 40}$$

$$\boldsymbol{a = 52 - 40 = 12}$$

$$\boldsymbol{\begin{aligned} 52 \cdot 28 &= (40 + 12)(40 - 12) \\ &= 40^2 - 12^2 = 1600 - 144 = 1456 \end{aligned}}$$

I can rewrite the product $52 \cdot 28$ as the difference of two squares, $x^2 - a^2$, where x is the arithmetic mean of the factors (numbers multiplied) and a is the positive difference between either factor and x. This means that $x = 40$ and $a = 12$.

2. Rewrite $121 - 36$ as a product of two integers.

$$\boldsymbol{\begin{aligned} 121 - 36 &= 11^2 - 6^2 \\ &= (11 + 6)(11 - 6) = 17 \cdot 5 \end{aligned}}$$

I know that $x^2 - a^2 = (x + a)(x - a)$. In this case, $x = 11$ and $a = 6$.

3. Quickly compute the difference of squares $84^2 - 16^2$.

$$\boldsymbol{84^2 - 16^2 = (84 - 16)(84 + 16) = 68 \cdot 100 = 6800}$$

Use Polynomial Identities to Determine if a Number is Prime

4. Is 1729 prime? Use the fact that $12^3 = 1728$ and an identity to support your answer.

$$\boldsymbol{\begin{aligned} 1729 &= 1728 + 1 \\ &= 12^3 + 1^3 \\ &= (12 + 1)\left(12^2 - (1)(12) + 1^2\right) \\ &= (13)(144 - 12 + 1) \\ &= 13 \cdot 133 \end{aligned}}$$

I know that $x^3 + a^3 = (x + a)(x^2 - ax + a)$. In this case, $x = 12$ and $a = 1$.

The number 1729 is not prime because it can be factored as $13 \cdot 133$.

5. Show that 99,999,951 is not prime without using a calculator or computer.

$$\begin{aligned}\mathbf{99,999,951} &= \mathbf{100,000,000 - 49}\\ &= \mathbf{10^8 - 49 = \left(10^4\right)^2 - 7^2}\\ &= \mathbf{\left(10^4+7\right)\left(10^4-7\right) = 10,007 \cdot 9993}\end{aligned}$$

I need to rewrite 99,999,951 either as a difference of perfect squares or as a sum or difference of perfect cubes.

Use Polynomial Identities to Determine Divisibility

6. Find a value of b so that the expression $b^n - 1$ is always divisible by 11 for any positive integer n. Explain why your value of b works for any positive integer n.

We can factor $b^n - 1$ as follows:

$$\boldsymbol{b^n - 1 = (b-1)\left(b^{n-1} + b^{n-2} + b^{n-3} + \cdots + b + 1\right).}$$

Choose $b = 12$ so that $b - 1 = 11$.

Since $b - 1 = 11$, then $b^n - 1$ will have 11 as a factor and therefore will always be divisible by 11.

I need to find a value b so 11 is a factor of $b^n - 1$. I can do this by finding a value of b so that $(b-1)$ is divisible by 11. One way to do this is to set $b - 1 = 11$, so that $b = 12$.

Note: Any value of b where $(b-1)$ is a multiple of 11 will produce a valid solution. Possible values of b include $12, 23, 34$, and 45.

7. Find a value of b so that the expression $b^n - 1$ is divisible by both 3 and 11 for any positive integer n. Explain why your value of b works for any positive integer n.

We can factor $b^n - 1$ as follows:

$$\boldsymbol{b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + b^{n-3} + \cdots + b + 1).}$$

Choose $b = 34$ so that $b - 1 = 33$.

Since $b - 1 = 33$, then $b^n - 1$ will have 33 as a factor, which is a multiple of both 3 and 11. Therefore, $b^n - 1$ will always be divisible by 3 and 11.

Note: Any value of b where $(b-1)$ is a multiple of 33 will produce a valid solution. Possible values of b include $34, 67$, and 100

Lesson 8: The Power of Algebra—Finding Primes

Apply Polynomial Identities to Factor Composite Numbers

1. Factor $5^{12}-1$ in three different ways: using the identity $(x^n-a^n)(x^{n-1}+ax^{n-2}+\cdots+a^{n-1})$, the difference of squares identity, and the difference of cubes identity.

i. $\mathbf{5^{12}-1=(5-1)\left(5^{11}+5^{10}+5^9+5^8+\cdots+5^1+1\right)}$

ii. $\mathbf{5^{12}-1=\left(5^6\right)^2-1^2}$

$\mathbf{=(5^6-1)(5^6+1)}$

I know that $5^{12}-1$ can be written as a difference of squares because it can be written in the form $x^{2n}-a^{2n}$, where $x=5$, $a=1$ and $n=6$.

I can then apply the difference of squares identity:

$$x^{2n}-a^{2n}=(x^n-a^n)(x^n+a^n).$$

iii. $\mathbf{5^{12}-1=\left(5^4\right)^3-1^3}$

$\mathbf{=\left(5^4-1\right)\left(\left(5^4\right)^2+1\left(5^4\right)+1^2\right)}$

$\mathbf{=\left(5^4-1\right)(5^8+5^4+1)}$

I know that $5^{12}-1$ can be written as a difference of cubes because it can be written in the form $x^{3n}-a^{3n}$, where $x=5$, $a=1$ and $n=4$.

I can then apply the difference of cubes identity:

$$x^{3n}-a^{3n}=(x^n-a^n)(x^{2n}+a^nx^n+a^{2n})$$

Apply Polynomial Identities to Analyze Divisibility

2. Explain why if n is odd, the number 3^n+1 will always be divisible by 4.

For any odd value of n,

For odd numbers n, I can use the identity $x^n+a^n=(x+a)(x^{n-1}-ax^{n-2}+a^2x^{n-3}-\cdots+a^{n-1})$.

$$\mathbf{3^n+1=3^n+a^n}$$
$$\mathbf{=(3+1)\left(3^{n-1}-(1)3^{n-2}+\left(1^2\right)3^{n-3}-\cdots+\left(1^{n-3}\right)3^2-\left(1^{n-2}\right)3+1^{n-1}\right)}$$
$$\mathbf{=4\left(3^{n-1}-(1)3^{n-2}+\left(1^2\right)3^{n-3}-\cdots+\left(1^{n-3}\right)3^2-\left(1^{n-2}\right)3+1^{n-1}\right).}$$

Since 4 is a factor of 3^n+1 for any odd value of n, the number 3^n+1 is divisible by 4.

3. If n is a composite number, explain why $4^n - 1$ is never prime.

Since n is composite, there are integers a and b larger than 1 so that $n = ab$. Then

I can use the identity
$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1)$.

$$\begin{aligned} \mathbf{4^n - 1} &= \mathbf{4^{ab} - 1} \\ &= \mathbf{(4^a - 1)\left((4^a)^{b-1} + (4^a)^{b-2} + (4^a)^{b-3} + \cdots + (4^a)^1 + 1\right)}. \end{aligned}$$

Since $a > 1$, the expression $4^a - 1$ must be an integer greater than or equal to 15, so $4^{ab} - 1$ has a factor other than 1. Therefore, the number $4^n - 1$ is composite.

Apply the Difference of Squares Identity to Rewrite Numbers

4. Express the numbers from 31 to 40 as the difference of two squares, if possible. The first four have been done for you.

Number	Factorization	Difference of two squares
31	$31 = 31 \cdot 1$ $= (16+15)(16-15)$	$16^2 - 15^2 = 256 - 225 = 31$
32	$32 = 8 \cdot 4$ $= (6+2)(6-2)$	$6^2 - 2^2 = 36 - 4 = 32$
33	$33 = 11 \cdot 3$ $= (7+4)(7-4)$	$7^2 - 4^2 = 49 - 16 = 33$
34	Can't be done.	
35	$\mathbf{35 = 7 \cdot 5}$ $\mathbf{= (6+1)(6-1)}$	$\mathbf{6^2 - 1^2 = 36 - 1 = 35}$
36	$\mathbf{36 = 6 \cdot 6}$ $\mathbf{= (6+0)(6-0)}$	$\mathbf{6^2 - 0^2 = 36 - 0 = 36}$
37	$\mathbf{37 = 37 \cdot 1}$ $\mathbf{= (19+18)(19-18)}$	$\mathbf{19^2 - 18^2 = 361 - 324 = 37}$
38	***Can't be done***	
39	$\mathbf{39 = 13 \cdot 3}$ $\mathbf{= (8+5)(8-5)}$	$\mathbf{8^2 - 5^2 = 64 - 25 = 39}$
40	$\mathbf{40 = 10 \cdot 4}$ $\mathbf{= (7+3)(7-3)}$	$\mathbf{7^2 - 3^2 = 49 - 9 = 40}$

I need to write each number as a product of two whole numbers. I can rewrite the product as a difference of two squares using the method we learned in Lesson 7.

If I cannot factor the number into two positive integers with an even sum, the number cannot be written as a difference of squares because the mean would be a fraction.

Lesson 9: Radicals and Conjugates

Convert Expressions to Simplest Radical Form

Express each of the following as a rational expression or in simplest radical form. Assume that the symbol x represents a positive number. Remember that a simplified radical expression has a denominator that is an integer.

1. $\sqrt{5x}\left(\sqrt{10}-\sqrt{2x}\right)$

$$\begin{aligned}\sqrt{5x}\cdot\sqrt{10}-\sqrt{5x}\cdot\sqrt{2x} &= \sqrt{50x}-\sqrt{10x^2}\\ &= \sqrt{25\cdot 2x}-\sqrt{10\cdot x^2}\\ &= \sqrt{25}\cdot\sqrt{2x}-\sqrt{10}\cdot\sqrt{x^2}\\ &= 5\sqrt{2x}-x\sqrt{10}\end{aligned}$$

I can distribute the $\sqrt{5x}$ to each term in the parentheses by multiplying the radicands.

Once I distribute the $\sqrt{5x}$ term to each term in the parentheses, I can factor each radicand so that one factor is a perfect square. The perfect square factors can be taken outside the radical.

2. $\sqrt[3]{\frac{2}{27}}+\sqrt[3]{24}-\sqrt[3]{\frac{1}{4}}$

$$\begin{aligned}\sqrt[3]{\frac{2}{27}}+\sqrt[3]{24}-\sqrt[3]{\frac{1}{4}} &= \frac{\sqrt[3]{2}}{\sqrt[3]{27}}+\sqrt[3]{8\cdot 3}-\frac{\sqrt[3]{1}}{\sqrt[3]{4}}\\ &= \frac{\sqrt[3]{2}}{\sqrt[3]{27}}+\sqrt[3]{8}\cdot\sqrt[3]{3}-\frac{\sqrt[3]{1\cdot 2}}{\sqrt[3]{4\cdot 2}}\\ &= \frac{\sqrt[3]{2}}{3}+2\sqrt[3]{3}-\frac{\sqrt[3]{2}}{\sqrt[3]{8}}\\ &= \frac{\sqrt[3]{2}}{3}+2\sqrt[3]{3}-\frac{\sqrt[3]{2}}{2}\\ &= \frac{2\sqrt[3]{2}}{6}+2\sqrt[3]{3}-\frac{3\sqrt[3]{2}}{6}\\ &= 2\sqrt[3]{3}-\frac{\sqrt[3]{2}}{6}\end{aligned}$$

I know that I can rewrite $\sqrt[3]{\frac{a}{b}}$ as $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$, which will allow me to simplify the numerator and denominator separately.

To write the denominator as an integer, I can multiply the numerator and denominator by a common factor that makes the radicand in the denominator a perfect cube.

Simplify Radical Quotients

Simplify each of the following quotients as far as possible.

3. $\frac{\sqrt[3]{12}-\sqrt[3]{3}}{\sqrt[3]{3}}$

$$\begin{aligned}\frac{\left(\sqrt[3]{12}-\sqrt[3]{3}\right)}{\sqrt[3]{3}} &= \frac{\sqrt[3]{12}}{\sqrt[3]{3}}-\frac{\sqrt[3]{3}}{\sqrt[3]{3}}\\ &= \sqrt[3]{\frac{12}{3}}-\sqrt[3]{\frac{3}{3}}\\ &= \sqrt[3]{4}-1\end{aligned}$$

I remember that an expression is in simplified radical form if it has no terms with an exponent greater than or equal to the index of the radical and if it does not have a radical in the denominator.

4. $\frac{5\sqrt{2}-\sqrt{3}}{2\sqrt{3}-4\sqrt{5}}$

$$\begin{aligned}\frac{5\sqrt{2}-\sqrt{3}}{2\sqrt{3}-4\sqrt{5}}\cdot\frac{2\sqrt{3}+4\sqrt{5}}{2\sqrt{3}+4\sqrt{5}} &= \frac{\left(5\sqrt{2}-\sqrt{3}\right)\cdot\left(2\sqrt{3}+4\sqrt{5}\right)}{\left(2\sqrt{3}-4\sqrt{5}\right)\cdot\left(2\sqrt{3}+4\sqrt{5}\right)}\\ &= \frac{10\sqrt{6}+20\sqrt{10}-2\sqrt{9}-4\sqrt{15}}{(2\sqrt{3})^2-(4\sqrt{5})^2}\\ &= \frac{10\sqrt{6}+20\sqrt{10}-2(3)-4\sqrt{15}}{12-80}\\ &= \frac{2\left(5\sqrt{6}+10\sqrt{10}-3-2\sqrt{15}\right)}{-68}\\ &= \frac{5\sqrt{6}+10\sqrt{10}-3-2\sqrt{15}}{-34}\\ &= \frac{3+2\sqrt{15}-5\sqrt{6}-10\sqrt{10}}{34}\end{aligned}$$

I can convert the denominator to an integer by multiplying both the numerator and denominator of the fraction by the radical conjugate.

Since $2\sqrt{3}-4\sqrt{5}$ is the expression, $2\sqrt{3}+4\sqrt{5}$ is its radical conjugate.

I know that for any positive numbers a and b, $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2$. In this case, $a=\left(2\sqrt{3}\right)^2=12$ and $b=\left(4\sqrt{5}\right)^2=80$.

Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Use the difference of squares identity $x^2 - y^2 = (x - y)(x + y)$ to simplify algebraic expressions.

1. Use the difference of squares identity to find $(x - y - 1)(x + y + 1)$.

$$\begin{aligned}(x - y - 1)(x + y + 1) &= \big(x - (y + 1)\big)\big(x + (y + 1)\big)\\ &= x^2 - (y + 1)^2\\ &= x^2 - \big(y^2 + 2y + 1\big)\\ &= x^2 - y^2 - 2y - 1\end{aligned}$$

I need to write the expression in the form $(a + b)(a - b)$.

Since both the y-term and the constant, 1, switch signs between the factors, I know that the b component of the identity is $y + 1$.

Use the difference of squares identity to factor algebraic expressions.

2. Use the difference of two squares identity to factor the expression $(x + y)(x - y) - 10x + 25$.

$$\begin{aligned}(x + y)(x - y) - 10x + 25 &= x^2 - y^2 - 10x + 25\\ &= \big(x^2 - 10x + 25\big) - y^2\\ &= (x - 5)^2 - y^2\\ &= \big((x - 5) - y\big)\big((x - 5) + y\big)\end{aligned}$$

I see that the constant term, 25, is a perfect square and that the coefficient of the linear x term is 10, which is twice the square root of 25. That reminds me of the identity:

$(x - a)^2 = x^2 - 2ax + a^2$.

Apply the difference of squares identity to find Pythagorean triples.

3. Prove that the Pythagorean triple $(15, 8, 17)$ can be found by choosing a pair of integers x and y with $x > y$ and computing $(x^2 - y^2, 2xy, x^2 + y^2)$.

We want $2xy = 8$, so $xy = 4$, so we can set $x = 4$ and $y = 1$.

I know $x = 4$ and $y = 1$ because $xy = 4$ and $x > y$.

This means that $x^2 - y^2 = 4^2 - 1^2 = 15$ and $x^2 + y^2 = 4^2 + 1^2 = 17$.

Therefore, the values $x = 4$ and $y = 1$ generate the Pythagorean triple

$$\left(x^2 - y^2, 2xy, x^2 + y^2\right) = (15, 8, 17).$$

4. Prove that the Pythagorean triple $(15, 36, 39)$ cannot be found by choosing a pair of integers x and y with $x > y$ and computing $(x^2 - y^2, 2xy, x^2 + y^2)$.

Since 36 is the only even number in $(15, 36, 39)$, we must have $2xy = 36$, so $xy = 18$.

I need to try all the cases where $xy = 18$ and $x > y$.

- ***If $x = 18$ and $y = 1$, then $x^2 - y^2 = 18^2 - 1^2 = 324 - 1 = 323$. So this combination does not produce a number in the triple.***
- ***If $x = 9$ and $y = 2$, then $x^2 - y^2 = 9^2 - 2^2 = 81 - 4 = 77$. So this combination does not produce a number in the triple.***
- ***If $x = 6$ and $y = 3$, then $x^2 - y^2 = 6^2 - 3^2 = 36 - 9 = 27$. So this combination does not produce a number in the triple.***

There are no other integer values of x and y that satisfy $xy = 18$ and $x > y$. Therefore, the triple $(15, 36, 39)$ cannot be found using the method described.

Lesson 11: The Special Role of Zero in Factoring

Find Solutions to Polynomial Equations

Find all solutions to the given equations.

1. $2x(x+1)^2(3x-4)=0$

   $\mathbf{2x=0}$ **or** $\mathbf{(x+1)=0}$ ***or*** $\mathbf{(3x-4)=0}$

   Solutions: $\mathbf{0, -1, \frac{4}{3}}$

The exponent 2 means that the equation has a repeated factor, which corresponds to a repeated solution.

To find all the solutions, I need to set each factor equal to zero and solve the resulting equations.

2. $x(x^2-25)(x^2-1)=0$

   $\mathbf{x(x-5)(x+5)(x-1)(x+1)=0}$

   $\mathbf{x=0}$ ***or*** $\mathbf{(x-5)=0}$ ***or*** $\mathbf{(x+5)=0}$ ***or*** $\mathbf{(x-1)=0}$ ***or*** $\mathbf{(x+1)=0}$

   Solutions: $\mathbf{0, 5, -5, 1, -1}$

3. $(x+3)^2=(3x-6)^2$

   $\mathbf{(x+3)^2-(3x-6)^2=0}$

   $\mathbf{\big((x+3)+(3x-6)\big)\big((x+3)-(3x-6)\big)=0}$

   $\mathbf{(4x-3)(-2x+9)=0}$

   $\mathbf{4x-3=0}$ ***or*** $\mathbf{-2x+9=0}$

   Solutions: $\mathbf{\frac{3}{4}, \frac{9}{2}}$

I recognize the expression on the left side of the equation as the difference of two perfect squares. I know that $a^2-b^2=(a+b)(a-b)$. In this case, $a=(x+3)$ and $b=(3x-6)$.

I collected like terms to rewrite the factors.

Determine Zeros and Multiplicity for Polynomial Functions

4. Find the zeros with multiplicity for the function $p(x)=(x^3-1)(x^4-9x^2)$.

   The number of times a solution appears as a factor in a polynomial function is its multiplicity.

I factored the expression (x^3-1) using the difference of cubes pattern.

Since (x^2+x+1) does not factor, it does not contribute any zeros of the function p.

   $\mathbf{0=(x-1)(x^2+x+1)x^2(x^2-9)}$

   $\mathbf{0=(x-1)(x^2+x+1)x^2(x+3)(x-3)}$

   Then $\mathbf{1}$ ***is a zero of multiplicity*** $\mathbf{1}$***,*** $\mathbf{0}$ ***is a zero of multiplicity*** $\mathbf{2}$***,*** $\mathbf{-3}$ ***is a zero of multiplicity*** $\mathbf{1}$***, and*** $\mathbf{3}$ ***is a zero of multiplicity*** $\mathbf{1}$***.***

2015-16

Construct a Polynomial Function That Has a Specified Set of Zeros with Stated Multiplicity

5. Find two different polynomial functions that have a zero at 1 of multiplicity 3 and a zero at -2 of multiplicity 2.

$\boldsymbol{p(x) = (x-1)^3(x+2)^2}$

$\boldsymbol{q(x) = 5(x-1)^3(x+2)^2}$

I need to include the linear factor $(x-1)$ three times and the linear factor $(x+2)$ twice in the statement of each function.

Compare the Remainder of $p(x) \div (x-a)$ with $p(a)$

6. Consider the polynomial function $p(x) = x^3 + x^2 + x - 6$.

   a. Divide p by the divisor $(x-2)$, and rewrite in the form $p(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.

   Either the division algorithm or the reverse tabular method can be used to find the quotient and remainder.

$$\begin{array}{r|rrrr}
 & & x^2 & +3x & +7 \\
\hline
x-2 & x^3 & +x^2 & +x & -6 \\
 & -(x^3 & -2x^2) & & \\
\cline{2-3}
 & & 3x^2 & +x & \\
 & & -(3x^2 & -6x) & \\
\cline{3-4}
 & & & 7x & -6 \\
 & & & -(7x & -14) \\
\cline{4-5}
 & & & & 8
\end{array}$$

I used the division algorithm we learned in Lesson 4 to find the quotient and remainder.

$\boldsymbol{p(x) = (x-2)\left(x^2+3x+7\right)+8}$

   b. Evaluate $p(2)$.

$$\begin{aligned}
&p(x) = (x-2)\left(x^2+3x+7\right)+8 \\
&p(2) = (2-2)\left(2^2+3\cdot 2+7\right)+8 \\
&p(2) = 0+8 \\
&p(2) = 8
\end{aligned}$$

Lesson 12: Overcoming Obstacles In Factoring

Solving Problems by Completing the Square

Solve each of the following equations by completing the square.

1. $x^2 - 8x + 3 = 0$

$$\mathbf{x^2 - 8x = -3}$$
$$\mathbf{x^2 - 8x + (-4)^2 = -3 + (-4)^2}$$
$$\mathbf{x^2 - 8x + 16 = -3 + 16}$$
$$\mathbf{(x-4)^2 = 13}$$
$$\mathbf{x - 4 = \pm\sqrt{13}}$$

I squared half the linear coefficient and added the result to both sides of the equation.

I recognize that $x^2 - 8x + 16$ fits the pattern for the square of a difference: $x^2 - 2ax + a^2 = (x-a)^2$

The solutions are $\mathbf{4+\sqrt{13}}$ and $\mathbf{4-\sqrt{13}}$.

2. $2x^2 - 6x = -1$

$$\mathbf{4x^2 - 12x = -2}$$
$$\mathbf{4x^2 - 12x + 9 = -2 + 9}$$
$$\mathbf{(2x-3)^2 = 7}$$
$$\mathbf{2x - 3 = \pm\sqrt{7}}$$

The solutions are $\mathbf{\frac{3+\sqrt{7}}{2}}$ and $\mathbf{\frac{3-\sqrt{7}}{2}}$.

I multiplied both sides of the equation by 2 so the quadratic term would be a perfect square. Then I can complete the square using the reverse tabular method, shown to the right.

	$\mathbf{2x}$	$\mathbf{-3}$
$\mathbf{2x}$	$\mathbf{4x^2}$	$\mathbf{-6x}$
$\mathbf{-3}$	$\mathbf{-6x}$	$\mathbf{9}$

3. $x^4 - 6x^2 + 4 = 0$

$$\mathbf{x^4 - 6x^2 = -4}$$
$$\mathbf{x^4 - 6x^2 + 9 = -4 + 9}$$
$$\mathbf{(x^2-3)^2 = 5}$$
$$\mathbf{x^2 - 3 = \pm\sqrt{5}}$$
$$\mathbf{x^2 = 3 \pm \sqrt{5}}$$

The solutions are $\mathbf{\sqrt{3+\sqrt{5}}}$, $\mathbf{\sqrt{3-\sqrt{5}}}$, $\mathbf{-\sqrt{3+\sqrt{5}}}$, and $\mathbf{-\sqrt{3-\sqrt{5}}}$.

Solving Problems by Using the Quadratic Formula

Solve the equation by using the quadratic formula.

4. $(x^2 - 7x + 3)(x^2 - 5x + 2) = 0$

$x = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(3)}}{2(1)}$ ***or*** $x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)}$

The solutions are $\frac{7+\sqrt{37}}{2}$, $\frac{7-\sqrt{37}}{2}$, $\frac{5+\sqrt{17}}{2}$, ***and*** $\frac{5-\sqrt{17}}{2}$.

> I can find the solutions by setting each factor equal to zero. I can solve the resulting quadratic equations using the quadratic formula:
> If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Solving Polynomial Equations by Factoring

Use factoring to solve each equation.

5. $6x^2 + 7x - 3 = 0$

$$\begin{aligned} \mathbf{6x^2 + 9x - 2x - 3} &= \mathbf{0} \\ \mathbf{3x(2x+3) - 1(2x+3)} &= \mathbf{0} \\ \mathbf{(3x-1)(2x+3)} &= \mathbf{0} \end{aligned}$$

The solutions are $-\frac{3}{2}$ ***and*** $\frac{1}{3}$.

> I split the linear term into two terms whose coefficients multiply to the product of the leading coefficient and the constant $(9 \cdot -2) = (6 \cdot -3)$. Then I can factor the resulting quadratic expression by grouping.

6. $(2x - 1)(x - 4) = (2x - 1)(x + 3)$

$$\begin{aligned} \mathbf{(2x-1)(x-4) - (2x-1)(x+3)} &= \mathbf{0} \\ \mathbf{(2x-1)\big((x-4) - (x+3)\big)} &= \mathbf{0} \\ \mathbf{(2x-1)(-7)} &= \mathbf{0} \end{aligned}$$

The only solution is $\frac{1}{2}$.

> I factored $(2x - 1)$ from both expressions.

7. $(x^2 + 5)^2 - 3 = 0$

$$\mathbf{\left((x^2+5) - \sqrt{3}\right)\left((x^2+5) + \sqrt{3}\right) = 0}$$

$$\mathbf{\left((x^2+5) - \sqrt{3}\right) = 0} \textit{ or } \mathbf{\left((x^2+5) + \sqrt{3}\right) = 0}$$

$$\mathbf{x^2 = -5 + \sqrt{3}} \textit{ or } \mathbf{x^2 = -5 - \sqrt{3}}$$

The solutions are $\sqrt{-5+\sqrt{3}}$, $-\sqrt{-5+\sqrt{3}}$, $\sqrt{-5-\sqrt{3}}$, ***and*** $-\sqrt{-5-\sqrt{3}}$.

> I factored the left side of the equation as a difference of two perfect squares: $a^2 - b^2 = (a + b)(a - b)$.

Lesson 13: Mastering Factoring

Factor Polynomials

1. If possible, factor the following polynomials over the real numbers.

a. $9x^2y^2 + 6xy - 8$

$$\begin{aligned}\mathbf{9(xy)^2 + 6(xy) - 8} &= \mathbf{9(xy)^2 + 12xy - 6xy - 8}\\ &= \mathbf{3xy(3xy+4) - 2(3xy+4)}\\ &= \mathbf{(3xy-2)(3xy+4)}\end{aligned}$$

Since the greatest common factor (GCF) among the terms is 1 and there are 3 terms, I split the middle term so its coefficients multiplied to the product of the leading coefficient and the constant: $12(-6) = 9(-8)$. I can then factor the polynomial by grouping just as we did in Lesson 12.

b. $18x^3 - 50xy^4z^6$

$$\begin{aligned}\mathbf{2x(9x^2 - 25y^4z^6)} &= \mathbf{2x\left((3x)^2 - (5y^2z^3)^2\right)}\\ &= \mathbf{2x(3x - 5y^2z^3)(3x + 5y^2z^3)}\end{aligned}$$

After factoring out the GCF, $2x$, I recognized the resulting expression was a difference of two perfect squares because the coefficients of the terms were perfect squares, and the exponents were even.

c. $3x^5 - 81x^2$

$$\mathbf{3x^2(x^3 - 27) = 3x^2(x-3)(x^2+3x+9)}$$

After factoring out the GCF, I recognize the factor $x^3 - 72$ as a difference of cubes.

d. $16x^4 + 25y^2$

Cannot be factored over the real numbers

This is the sum of two perfect squares, and I cannot factor the sum of squares over the real numbers.

e. $2x^4 - 50$

$$\begin{aligned} 2(x^4 - 25) &= 2\left((x^2)^2 - 5^2\right) \\ &= 2(x^2 + 5)(x^2 - 5) \\ &= 2(x^2 + 5)(x - \sqrt{5})(x + \sqrt{5}) \end{aligned}$$

2.

a. Factor $y^6 - 1$ first as a difference of cubes, and then factor completely: $(y^2)^3 - 1^3$.

$$\begin{aligned} y^6 - 1 &= (y^2 - 1)(y^4 + 1y^2 + 1) \\ &= (y - 1)(y + 1)(y^4 + y^2 + 1) \end{aligned}$$

When I factor the expression as a difference of cubes, the first expression can be factored as a difference of squares.

b. Factor $y^6 - 1$ first as a difference of squares, and then factor completely: $(y^3)^2 - 1^2$.

$$\begin{aligned} y^6 - 1 &= (y^3 - 1)(y^3 + 1) \\ &= (y - 1)(y^2 + y + 1)(y + 1)(y^2 - y - 1) \end{aligned}$$

When I factor the expression as a difference of squares, the first expression can be factored as a difference of cubes and the second as the sum of cubes.

c. Use your results from parts (a) and (b) to factor $y^4 + y^2 + 1$.

From part (a), we know $y^6 - 1 = (y - 1)(y + 1)(y^4 + y^2 + 1)$.
From part (b), we know $y^6 - 1 = (y - 1)(y^2 + y + 1)(y + 1)(y^2 - y - 1)$, ***which is equivalent to*** $y^6 - 1 = (y - 1)(y + 1)(y^2 + y + 1)(y^2 - y - 1)$.

It follows that $y^4 + y^2 + 1 = (y^2 + y + 1)(y^2 - y - 1)$.

Lesson 14: Graphing Factored Polynomials

Steps for Creating Sketches of Factored Polynomials

First, determine the zeros of the polynomial from the factors. These values are the x-intercepts of the graph of the function. Mark these values on the x-axis of a coordinate grid.

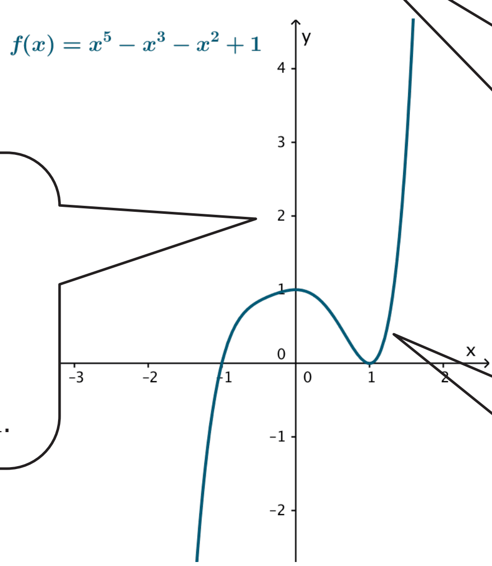
Evaluate the function at x-values between the zeros to determine whether the graph of the function is negative or positive between the zeros. If the output is positive, the graph lies above the x-axis. If the output is negative, the graph lies below the x-axis.

Determining the Number of x-intercepts and Relative Maxima and Minima for Graphs of Polynomials

1. For the function $f(x) = x^5 - x^3 - x^2 + 1$, identify the largest possible number of x-intercepts and the largest possible number of relative maxima and minima based on the degree of the polynomial. Then use a calculator or graphing utility to graph the function and find the actual number of x-intercepts and relative maxima and minima.

The largest number of possible x-intercepts is 5, and the largest possible number of relative maxima and minima is 4.

I know the largest possible number of x-intercepts is equal to the degree of the polynomial, and the largest possible number of relative maxima and relative minima is equal to one less than the degree of the polynomial.



I know $(0, 1)$ is a relative maximum point because the output values for inputs immediately greater than or less than 0 are less than 1.

I know $(1, 0)$ is a relative minimum point because the output values for inputs immediately to the right and left of 1 are greater than 0.

The graph of f has two x-intercepts, one relative maximum point, and one relative minimum point.

2015-16

Sketching a Graph from Factored Polynomials

2. Sketch a graph of the function $f(x) = 2(x+4)(x-1)(x+2)$ by finding the zeros and determining the sign of the values of the function between zeros.

The zeros are $-2, 1$, and -4.

For $x < -4$: $f(-5) = -36$, so the graph of f is below the x-axis for $x < -4$.

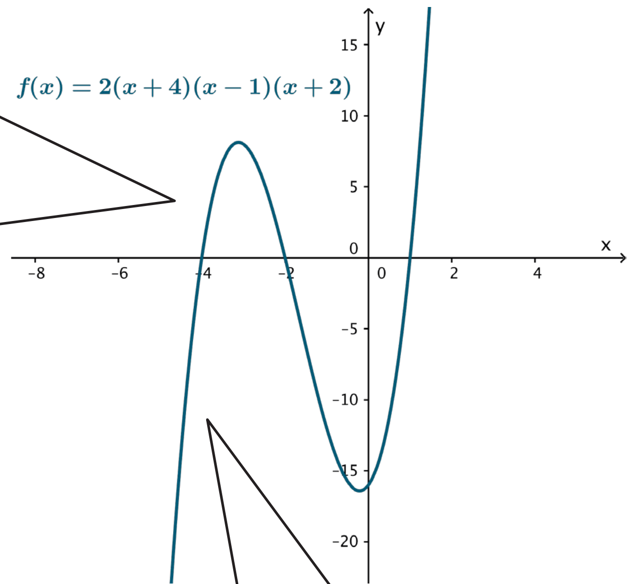
For $-4 < x < -2$: $f(-3) = 8$, so the graph of f is above the x-axis for $-4 < x < -2$.

For $-2 < x < 1$: $f(0) = -16$, so the graph of f is below the x-axis for $-2 < x < 1$.

For $x > 1$: $f(2) = 48$, so the graph is above the x-axis for $x > 1$.

If I substitute input values between the zeros, I can determine from the resulting outputs if the graph of f will be above or below the x-axis for the regions between the zeros.

Since there are no zeros for $-4 < x < -2$, I know the graph of f will not intersect the x-axis for inputs between -4 and -2. Then, since $f(-3)$ is positive, all outputs for $-4 < x < -2$ will also be positive, and this part of the graph will be above the x-axis.



Since -4 is the zero with the least value, I know the graph of f will not intersect the x-axis for inputs less than -4, which means that since $f(-5)$ is negative, all outputs for $x < -4$ will also be negative, and this part of the graph will be below the x-axis.

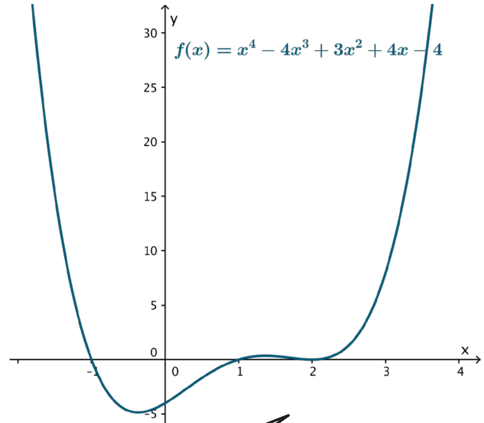
3. Sketch a graph of the function $f(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$ by determining the sign of the values of the function between zeros $-1, 1,$ and 2.

For $x < -1$: Since $f(-2) = 48$, the graph is above the x-axis for $x < -1$.

For $-1 < x < 1$: Since $f(0) = -4$, the graph is below the x-axis for $-1 < x < 1$.

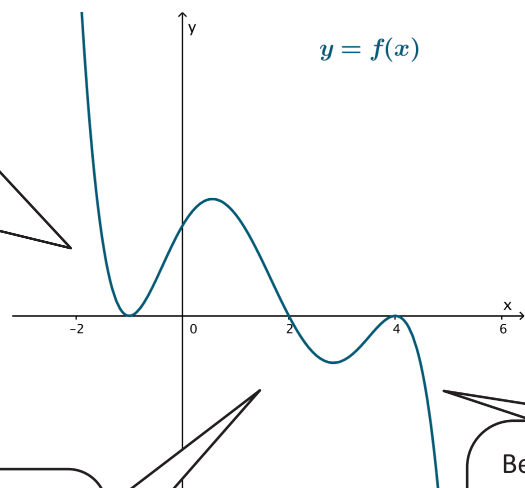
For $1 < x < 2$: Since $f(1.5) = 0.3125$, the graph is above the x-axis for $1 < x < 2$.

For $x > 2$: Since $f(3) = 8$, the graph is above the x-axis for $x > 2$.



I notice that the graph is above the x-axis on either side of the x-intercept at 2. This means that $(2, 0)$ is a relative minimum point for this function.

4. A function f has zeros at -1, 2, and 4. We know that $f(-2)$ and $f(0)$ are positive, while $f(3)$ and $f(5)$ are negative. Sketch a graph of f.



Because $f(-2)$ and $f(0)$ are positive and $f(-1) = 0$, I know the graph of f lies above the x-axis for $x < -1$ and $-1 < x < 2$ and that it intersects the x-axis at -1,

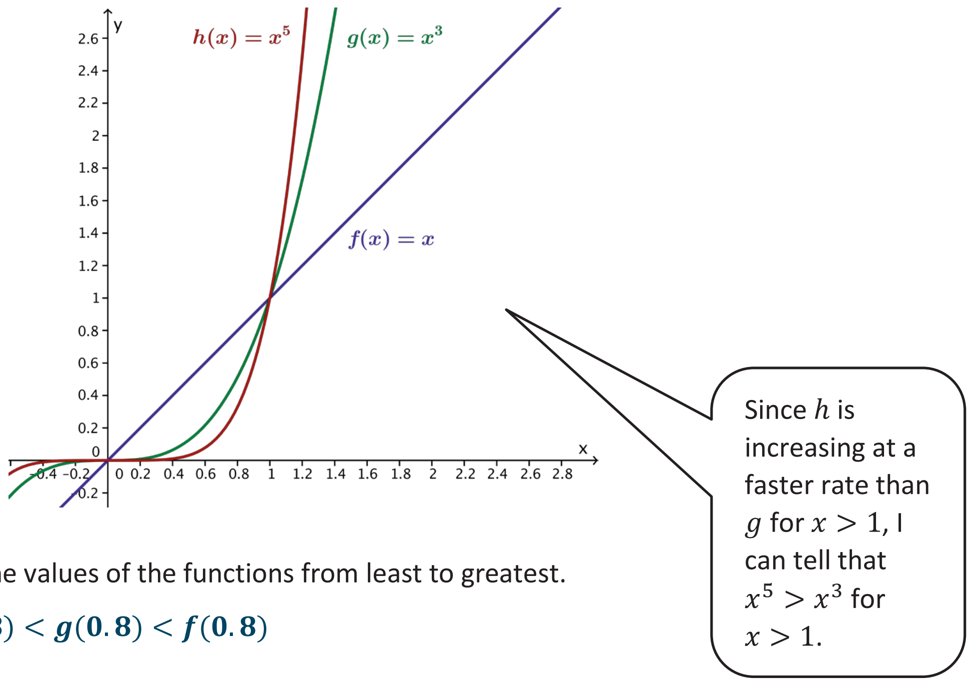
Because $f(0)$ is positive and $f(3)$ is negative and $f(2) = 0$, I know the graph of f crosses the x-axis at 2.

Because $f(3)$ and $f(5)$ are negative and $f(4) = 0$, I know the graph of f lies below the x-axis for $x > 4$ and $2 < x < 4$ and that it intersects the x-axis at 4.

Lesson 15: Structure in Graphs of Polynomial Functions

Characteristics of Polynomial Graphs

1. Graph the functions $f(x) = x$, $g(x) = x^3$, and $h(x) = x^5$ simultaneously using a graphing utility, and zoom in at the origin.



a. At $x = 0.8$, order the values of the functions from least to greatest.

At $x = 0.8$, $h(0.8) < g(0.8) < f(0.8)$

b. At $x = 2$, order the values of the functions from least to greatest.

At $x = 2$, $f(2) < g(2) < h(2)$

c. At $x = 1$, order the values of the functions from least to greatest.

At $x = 1$, $f(1) = g(1) = h(1)$

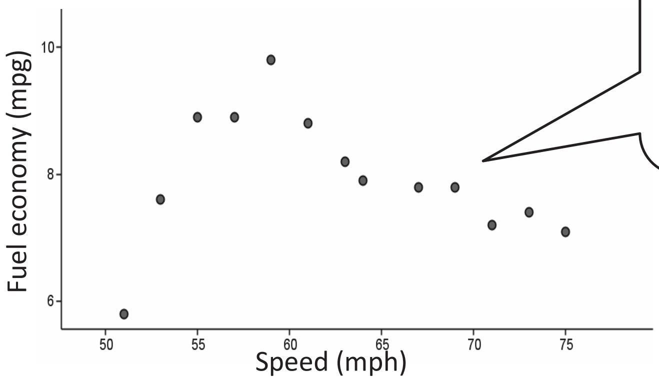
2. The Oak Ridge National Laboratory conducts research for the United States Department of Energy. The following table contains data from the ORNL about fuel efficiency and driving speed on level roads for vehicles that weigh between $60{,}000$ lb to $70{,}000$ lb.

Fuel Economy Versus Speed for Flat Terrain

Speed (mph)	Fuel Economy (mpg)
51	5.8
53	7.6
55	8.9
57	8.9
59	9.8
61	8.8
63	8.2
65	7.9
67	7.8
69	7.8
71	7.2
73	7.4
75	7.1

Source: www-cta.ornl.gov/cta/Publications/Reports/ORNL_TM_2011_471.pdf - 400k - 2011-12-05

a. Plot the data using a graphing utility.



The shape of the graph shows that both ends of the graph point downward, so I know that it would be best represented using a polynomial of even degree.

b. Determine if the data display the characteristics of an odd- or even-degree polynomial function.

The characteristics are similar to that of an even-degree polynomial function.

c. List one possible reason the data might have such a shape.

Fuel efficiency decreases at excessive speeds.

3. Determine if each of the following functions is even or odd. Explain how you know.

a. $f(x) = 3x^3 - 5x$

$\boldsymbol{f(-x) = 3(-x)^3 - 5(-x) = -3x^3 + 5x = -(3x^3 - 5x) = -f(x)}$

The function f is odd.

b. $f(x) = 2x^4 - 6x^2 + 4$

$\boldsymbol{f(-x) = 2(-x)^4 - 6(-x)^2 + 4 = 2x^4 - 6x^2 + 4 = f(x)}$

The function f is even.

I know that $f(-x) = f(x)$ for even functions and $f(-x) = -f(x)$ for odd functions. This function does not meet either condition.

c. $f(x) = x^3 + x^2 + 1$

$\boldsymbol{f(-x) = (-x)^3 + (-x)^2 + 1 = -x^3 + x^2 + 1}$

The function f is neither even nor odd.

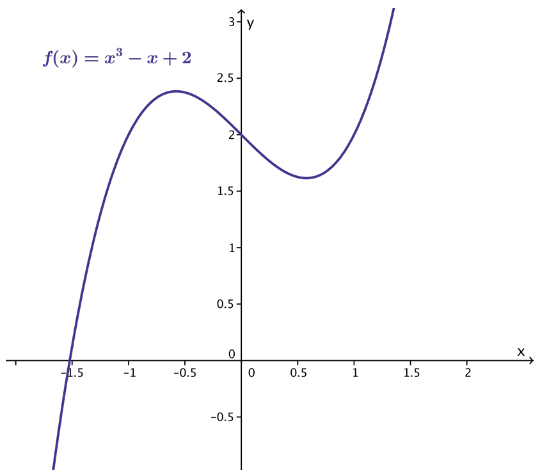
4. Determine the y-intercept for each function below.

a. $f(x) = x^3 - x + 2$

$\boldsymbol{f(0) = 0^3 - 0 + 2 = 2}$

The y-intercept is 2.

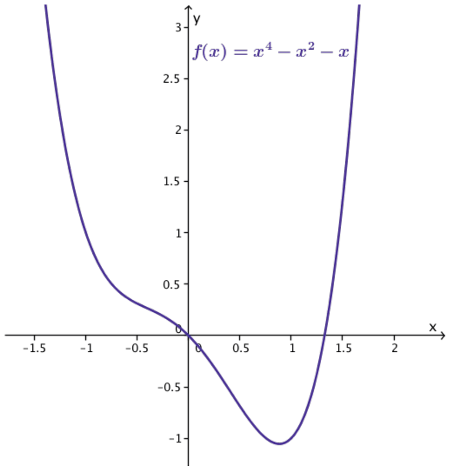
I know the y-intercept is the output of a function when the input is 0.



b. $f(x) = x^4 - x^2 - x$

$\boldsymbol{f(0) = 0^4 - 0^2 - 0 = 0}$

The y-intercept is 0.

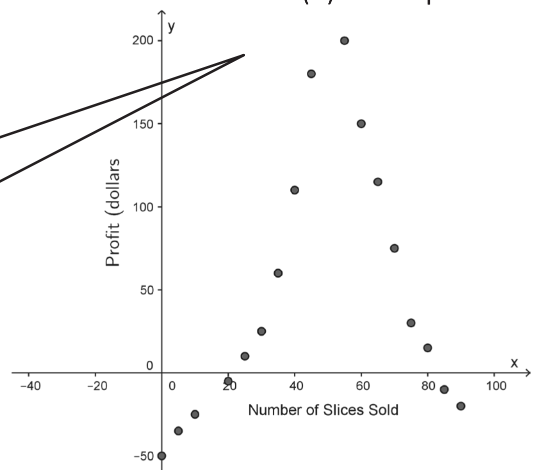


Lesson 16: Modeling with Polynomials—An Introduction

Modeling Real-World Situations with Polynomials

1. For a fundraiser, members of the math club decide to make and sell slices of pie on March 14. They are trying to decide how many slices of pie to make and sell at a fixed price. They surveyed student interest around school and made a scatterplot of the number of slices sold (x) versus profit in dollars (y).

With a single turning point, I recognize that this data can be modeled using a quadratic polynomial function.



a. Identify the x-intercepts and the y-intercept. Interpret their meaning in context.

The x-intercepts are approximately 20 and 82. The number 20 represents the number of slices of pie that would be made and sold for the math club to break even; in other words, the revenue would be equal to the amount spent on supplies. The number 82 represents the number of slices above which the supply would exceed the demand so that the cost to produce the slices of pie would exceed the revenue.

The y-intercept is approximately -50. The -50 represents the money that the math club members must spend on supplies in order to make the pies. That is, they will lose $\$50$ if they sell 0 slices of pie.

b. How many slices of pie should they sell in order to maximize the profit?

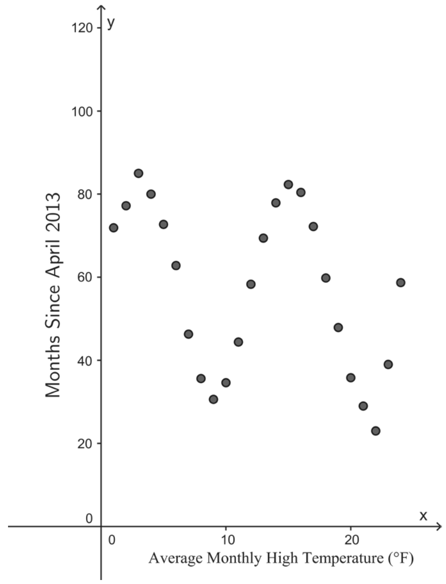
They should sell approximately 55 slices to maximize the profit.

From the graph, I can see that the coordinates of the vertex are approximately $(55, 200)$. The number 55 is the number of slices sold, and the number 200 is the profit, in dollars.

c. What is the maximum profit?

The maximum profit is slightly more than $\$200$.

2. The following graph shows the average monthly high temperature in degrees Fahrenheit (x) in Albany, NY, from May 2013 through April 2015, where y represents the number of months since April 2013. (Source: U.S. Climate Data)



a. What degree polynomial would be a reasonable choice to model this data?

Since the graph has 4 turning points (2 relative minima, 2 relative maxima), a degree 5 polynomial could be used.

I remember that the maximum number of relative maxima and minima is equal to one less than the degree of the polynomial.

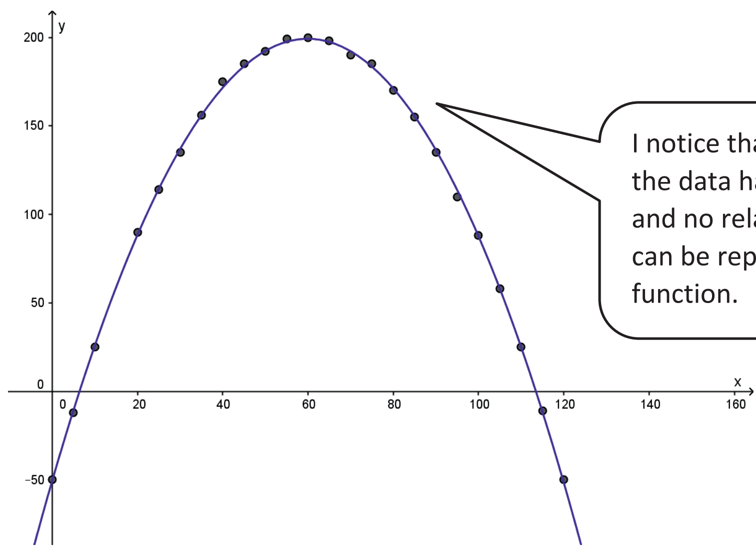
b. Let T be the function that represents the temperature, in degrees Fahrenheit, as a function of time x, in months. What is the value of $T(10)$, and what does it represent in the context of the problem?

The value $T(10)$ represents the average monthly high temperature ten months after April 2013, which is February 2014. From the graph, $T(10) \approx 35$, which means the average monthly high temperature in February 2014 in Albany was about $35°\text{F}$.

Lesson 17: Modeling with Polynomials—An Introduction

Modeling Real-World Situations with Polynomials

1. Recall the math club fundraiser from the Problem Set of the previous lesson. The club members would like to find a function to model their data, so Joe draws a curve through the data points as shown.



I notice that the curve used to model the data has one relative maximum and no relative minima, so the curve can be represented with a quadratic function.

a. The function that models the profit in terms of the number of slices of pie made has the form $P(x) = c(x^2 - 119x + 721)$. Use the vertical intercept labeled on the graph to find the value of the leading coefficient c.

Since $P(0) = -50$, we have $-50 = c\left(0^2 - 119(0) + 721\right)$.

Then $-50 = 721c$, so $c \approx -0.07$.

So $P(x) = -0.07(x^2 - 119x + 721)$ is a reasonable model.

I see from the graph that the vertical intercept is -50. I can substitute $(0, -50)$ into the function and isolate c to find its value.

b. From the graph, estimate the profit if the math club sells 50 slices of pie.

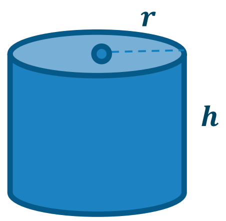
Reading from the graph, the profit is approximately $\$190$ if the club sells 50 slices of pie.

c. Use your function to estimate the profit if the math club sells 50 slices of pie.

Because $P(50) = -0.07\left((50)^2 - 119(50) + 721\right) = 191.03$, the equation predicts a profit of $\$191.03$.

2. A container is to be constructed as a cylinder with no top.

   a. Draw and label the sides of the container.



   b. The surface area is 150 cm^2. Write a formula for the surface area S, and then solve for h.

$$\boldsymbol{S = \pi r^2 + 2\pi r h = 150}$$

$$\boldsymbol{h = \frac{150 - \pi r^2}{2\pi r}}$$

I know the surface area of the container is the sum of the area of the circular base, πr^2, added to the lateral area of the cylinder. I can find the lateral area by multiplying the height of the cylinder by the circumference of the circular base, which is $2\pi rh$.

   c. Write a formula for the function of the volume of the container in terms of r.

$$\boldsymbol{V(r) = \pi r^2 h = \pi r^2\left(\frac{150 - \pi r^2}{2\pi r}\right) = \frac{150\pi r^2 - \pi^2 r^4}{2\pi r} = 75r - \frac{\pi r^3}{2}}$$

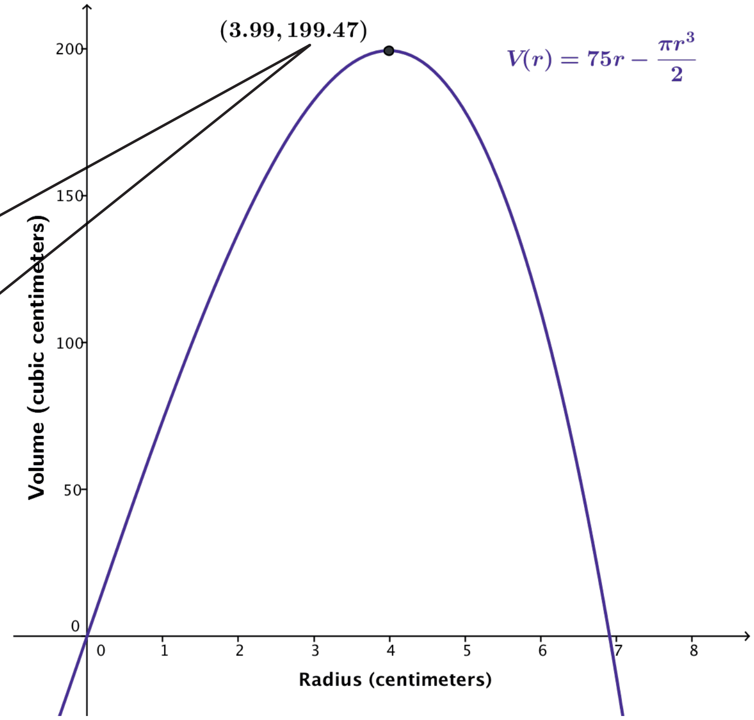
I know the volume of a cylinder is the product of the area of its circular base and the height.

I can substitute the expression for h from part (b) to write an expression for volume in terms of r.

d. Use a graph of the volume as a function of r to find the maximum volume of the container.

The maximum volume is approximately $\mathbf{199.47\ cm^3}$ ***when the radius is approximately*** $\mathbf{3.99\ cm}$.

I know the x-coordinate of the **relative maximum** point represents the value of r that maximizes volume. I can substitute this value into the expression for part (b) to find h.



e. What dimensions should the container have in order to maximize its volume?

The relative maximum occurs when $\boldsymbol{r \approx 3.99}$. ***Then*** $\boldsymbol{h = \frac{150-\pi r^2}{2\pi r} \approx 3.99}$. ***The container should have radius approximately*** $\mathbf{3.99\ cm}$ ***and height approximately*** $\mathbf{3.99\ cm}$ ***to maximize its volume.***

Lesson 18: Overcoming a Second Obstacle in Factoring—What If There Is a Remainder?

Finding a Quotient of Two Polynomials by Inspection

1. For the pair of problems shown, find the first quotient by factoring the numerator. Then, find the second quotient by using the first quotient.

$$\frac{x^2-x-12}{x-4}$$

$$\frac{\boldsymbol{x^2-x-12}}{\boldsymbol{x-4}} = \frac{\boldsymbol{(x-4)(x+3)}}{\boldsymbol{(x-4)}} = \boldsymbol{x+3}$$

Once I factor the numerator, I can see that the numerator and denominator have a common factor by which I can divide, if we assume that $x-4 \neq 0$.

$$\frac{x^2-x-15}{x-4}$$

$$\frac{\boldsymbol{x^2-x-15}}{\boldsymbol{x-4}} = \frac{\boldsymbol{(x^2-x-12)-3}}{\boldsymbol{(x-4)}} = \frac{\boldsymbol{(x^2-x-12)}}{\boldsymbol{(x-4)}} - \frac{\boldsymbol{3}}{\boldsymbol{(x-4)}} = \boldsymbol{x+3} - \frac{\boldsymbol{3}}{\boldsymbol{(x-4)}}$$

I can see that the numerator of this rational expression can be rewritten as the difference of the numerator of the first expression and 3, which means the quotient will be the same as in the first expression, but I will have a remainder of -3.

2. Rewrite the numerator in the form $(x-h)^2 + k$ by completing the square. Then, find the quotient.

$$\frac{x^2-2x-1}{x-1}$$

$$\begin{aligned}\frac{\boldsymbol{x^2-2x-1}}{\boldsymbol{x-1}} &= \frac{\boldsymbol{(x^2-2x)-1}}{\boldsymbol{x-1}}\\ &= \frac{\boldsymbol{(x^2-2x+1)-1-1}}{\boldsymbol{x-1}}\\ &= \frac{\boldsymbol{(x^2-2x+1)-2}}{\boldsymbol{x-1}}\\ &= \frac{\boldsymbol{x^2-2x+1}}{\boldsymbol{x-1}} - \frac{\boldsymbol{2}}{\boldsymbol{x-1}}\\ &= \frac{\boldsymbol{(x-1)^2}}{\boldsymbol{x-1}} - \frac{\boldsymbol{2}}{\boldsymbol{x-1}}\\ &= \boldsymbol{(x-1)} - \frac{\boldsymbol{2}}{\boldsymbol{x-1}}\end{aligned}$$

To complete the square with the expression x^2-2x, I need to add a constant that is the square of half the linear coefficient, which means I need to add 1. To maintain the value of the numerator x^2-2x-1, I can rewrite it as $(x^2-2x+1)-1-1$, and then I can rewrite the expression in the form $(x-h)^2 + k$.

Finding a Quotient of Two Polynomials by the Reverse Tabular Method

3. Find the quotient by using the reverse tabular method.

$$\frac{x^3-6x+1}{x-2}$$

This is the method I used to divide polynomials in Lesson 3.

	$\boldsymbol{x^2}$	$\boldsymbol{2x}$	$\boldsymbol{-2}$	
	$\boldsymbol{x^3}$	$\boldsymbol{2x^2}$	$\boldsymbol{-2x}$	$\boldsymbol{x}$
$\boldsymbol{x^3}$	$\boldsymbol{-2x^2}$	$\boldsymbol{-4x}$	$\boldsymbol{4}$	$\boldsymbol{-2}$
$\boldsymbol{0x^2}$	$\boldsymbol{-6x}$	$\boldsymbol{1 = 4 - 3}$		

The remainder is equal to the difference between the constant in the dividend and the constant in the table.

The remainder is $\boldsymbol{-3}$.

$$\frac{x^3-6x+1}{x-2}=\frac{x^3-6x+4-3}{x-2}=\frac{x^3-6x+4}{x-2}-\frac{3}{x-2}=x^2+2x-2-\frac{3}{x-2}$$

Finding a Quotient of Two Polynomials Using Long Division

4. Find the quotient by using long division.

This is the method I used to divide polynomials in Lesson 5.

$$\begin{array}{rrrrrrl}
 & x^3 & -3x^2 & +3x & +2 & -\dfrac{2}{x+1} \\
\cline{2-6}
x+1 \; \big) & x^4 & -2x^3 & +0x^2 & +5x & +0 \\
 & -(x^4 & +x^3) & & & \\
\cline{2-3}
 & & -3x^3 & +0x^2 & & \\
 & & -(-3x^3 & -3x^2) & & \\
\cline{3-4}
 & & & 3x^2 & +5x & \\
 & & & -(3x^2 & +3x) & \\
\cline{4-5}
 & & & & 2x & +0 \\
 & & & & -(2x & +2) \\
\cline{5-6}
 & & & & & -2
\end{array}$$

Using placeholders helps me to align like terms as I carry out the long division procedure.

Lesson 19: The Remainder Theorem

Applying the Remainder Theorem to Find Remainders and Evaluate Polynomials at Specific Inputs

1. Use the remainder theorem to find the remainder for the following division.

$$\frac{x^4 + 2x^3 - 6x + 1}{x + 3}$$

Let $\boldsymbol{P(x) = x^4 + 2x^3 - 6x + 1}$***. Then***

$$\boldsymbol{P(-3) = (-3)^4 + 2(-3)^3 - 6(-3) + 1}$$
$$\boldsymbol{= 81 - 54 + 18 + 1 = 46.}$$

Therefore, the remainder of the division $\boldsymbol{\frac{x^4+2x^3-6x+1}{x+3}}$ ***is*** $\boldsymbol{46}$***.***

The remainder theorem tells me that when polynomial P is divided by $(x - (-3))$, the remainder is the value of P evaluated at -3.

2. Consider the polynomial $P(x) = x^3 + x^2 - 5x + 2$. Find $P(3)$ in two ways.

$$\boldsymbol{P(3) = (3)^3 + (3)^2 - 5(3) + 2 = 23}$$

$$\boldsymbol{\frac{x^3 + x^2 - 5x + 2}{x - 3} = x^2 + 4x + 7 + \frac{23}{x - 3}}$$

so $\boldsymbol{P(3) = 23}$***.***

The remainder theorem tells me that $P(3)$ has the same value as the remainder to the division of P by $(x - 3)$.

3. Find the value k so that $\frac{kx^3-x+k}{x-1}$ has remainder 10.

Let $\boldsymbol{P(x) = kx^3 - x + k}$***. Then*** $\boldsymbol{P(1) = 10}$***, so***

$$\boldsymbol{k(1^3) - 1 + k = 10}$$
$$\boldsymbol{2k - 1 = 10}$$
$$\boldsymbol{k = \frac{11}{2}.}$$

Since the remainder theorem states that P divided by $(x - 1)$ has a remainder equal to $P(1)$, I know $P(1) = 10$.

Applying the Factor Theorem

4. Determine whether the following are factors of the polynomial $P(x) = x^3 - 4x^2 + x + 6$.

   a. $(x - 2)$

   $P(2) = (2)^3 - 4(2)^2 + (2) + 6 = 0$

   Yes, $x - 2$ is a factor of P.

   b. $(x - 1)$

   $P(1) = (1)^3 - 4(1)^2 + (1) + 6 = 4$

   No, $x - 1$ is not a factor of P.

By the factor theorem, I know that if $P(a) = 0$, then $(x - a)$ is a factor of P. By the remainder theorem, I know that if $(x - a)$ is a factor of P, then $P(a) = 0$. This means that if $P(a) \neq 0$, then $(x - a)$ is not a factor of P.

Using the Remainder Theorem to Graph Polynomials

5. Consider the polynomial function $P(x) = x^3 + 3x^2 - x - 3$.

   a. Verify that $P(-1) = 0$.

   $P(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3 = 0$

   b. Since $P(-1) = 0$, what are the factors of polynomial P?

   Since $P(-1) = 0$, $(x + 1)$ is a factor of P.

   Since $\frac{x^3+3x^2-x-3}{x+1} = x^2 + 2x - 3$, it follows that

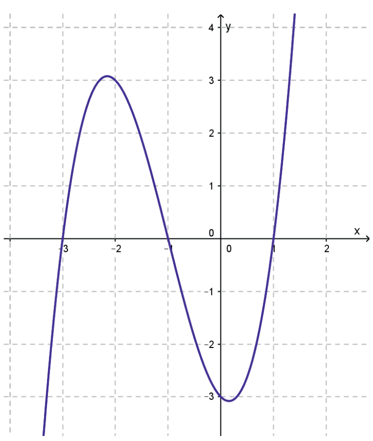
   $$x^3 + 3x^2 - x - 3 = (x + 1)(x^2 + 2x - 3) = (x + 1)(x + 3)(x - 1).$$

   The factors of P are $x + 1, x + 3$ and $x - 1$.

I can find this quotient using long division or the reverse tabular method. I can then factor this quotient as the product of two linear factors.

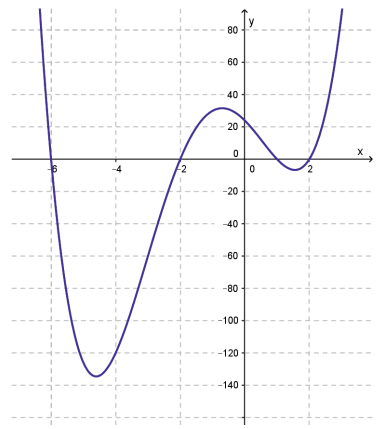
   c. Find the zeros of P, and then use the zeros to sketch the graph of P.

   The zeros of P are -1, -3, and 1.



I can substitute inputs into P whose values are between the values of the zeros to help discover the shape of the graph. For example, $P(0) = -3$, so the graph is beneath the x-axis between the x-values of -1 and 1.

Using the Factor Theorem to Write Equations for Polynomials



6. A fourth-degree polynomial function g is graphed at right.

   a. Write a formula for g in factored form using c for the constant factor.

   $\boldsymbol{f(x) = c(x+6)(x+2)(x-1)(x-2)}$

   Since f has zeros at $-6, -2, 1$, and 2, I know f has factors $(x+6)$, $(x+2)$, $(x-1)$, and $(x-2)$. I know that since f is a fourth-degree polynomial with four zeros, each zero has a multiplicity of 1.

   b. Use the fact that $f(-4) = -120$ to find the constant factor c.

   $\boldsymbol{-120 = c(-4+6)(-4+2)(-4-1)(-4-2)}$
   $\boldsymbol{-120 = -120c}$
   $\boldsymbol{c = 1}$
   $\boldsymbol{f(x) = (x+6)(x+2)(x-1)(x-2)}$

Lesson 20: Modeling Riverbeds with Polynomials

This lesson requires students to fit a polynomial function to data values in order to model a riverbed using a polynomial function.

1. Use the remainder theorem to find the polynomial P of least degree so that $P(3) = 1$ and $P(5) = -3$.

$\boldsymbol{P(x) = (x-3)q(x) + 1}$

$\boldsymbol{P(x) = (x-5)q(x) - 3}$

The remainder theorem tells me that for any number a, I can find some polynomial q so that $P = (x-a)q(x) + P(a)$.

$$\begin{aligned} (x-3)q(x)+1 &= (x-5)q(x)-3 \\ xq(x)-3q(x)+1 &= xq(x)-5q(x)-3 \\ 2q(x) &= -4 \\ q(x) &= -2 \end{aligned}$$

$\boldsymbol{P(x) = (x-3)(-2) + 1 = -2x + 7}$

Check: $\boldsymbol{P(3) = -2(3) + 7 = 1}$
$\boldsymbol{P(5) = -2(5) + 7 = -3}$.

2. Write a quadratic function P such that $P(0) = 4$, $P(2) = -4$, and $P(5) = -1$ using a system of equations.

$\boldsymbol{P(x) = ax^2 + bx + c}$

$\boldsymbol{c = 4}$

Since $P(0) = 4$ and the y-intercept of P is c, I know that $c = 4$.

$\boldsymbol{P(x) = ax^2 + bx + 4}$

$\boldsymbol{P(2) = a(2)^2 + b(2) + 4 = -4}$, ***so*** $\boldsymbol{4a + 2b = -8}$

$\boldsymbol{P(5) = a(5)^2 + b(5) + 4 = -1}$, ***so*** $\boldsymbol{25a + 5b = -5}$

I can evaluate P at 2 and at 5 and set the expressions that result equal to the known outputs to create a system of two linear equations.

I can solve the system using elimination.

$$\begin{aligned} 5(4a+2b) &= 5(-8) \quad \textit{so} \quad 20a + 10b = -40 \\ -2(25a+5b) &= -2(-5) \quad \textit{so} \quad -50a - 10b = 10 \end{aligned}$$

Adding the two equations gives $\boldsymbol{-30a = -30}$, ***so*** $\boldsymbol{a = 1}$.

$\boldsymbol{4(1) + 2b = -8}$, ***so*** $\boldsymbol{b = -6}$.

Once I found the value of a, I back-substituted its value into the first linear equation to find the value of b.

$\boldsymbol{P(x) = x^2 - 6x + 4}$

Representing Data with a Polynomial Using the Remainder Theorem

To use this strategy, remember that $P(x) = (x - a)q(x) + r$, where q is a polynomial with degree one less than P and r is the remainder. The remainder theorem tells us that $r = P(a)$.

3. Find a degree-three polynomial function P such that $P(-2) = 0$, $P(0) = 4$, $P(2) = 16$, and $P(3) = 55$. Use the table below to organize your work. Write your answer in standard form, and verify by showing that each point satisfies the equation.

Function Value	**Substitute the data point into the current form of the equation for P.**	**Apply the remainder theorem to a, b, or c.**	**Rewrite the equation for P in terms of a, b, or c.**
$P(-2) = 0$	Since $P(-2) = 0$, I know $(x+2)$ is a factor of P. Then there is some polynomial a so that $P(x) = (x+2)a(x)$.		$P(x) = (x+2)a(x)$
$P(0) = 4$	$P(x) = (x+2)a(x)$ $P(0) = (0+2)a(0)$ $4 = 2a(0)$ $a(0) = 2$ The remainder theorem tells me $a(x) = (x-0)b(x) + a(0)$.	$a(x) = xb(x) + 2$	$P(x) = (x+2)(xb(x)+2)$ I can replace $a(x)$ with the equivalent expression $xb(x) + 2$.
$P(2) = 16$	$P(x) = (x+2)(xb(x)+2)$ $P(2) = (2+2)(2b(2)+2)$ $16 = 4(2b(2)+2)$ $4 = 2b(2)+2$ $b(2) = 1$	$b(x) = (x-2)c(x) + 1$	$P(x) = (x+2)[x[(x-2)c(x)+1]+2]$
$P(3) = 55$	$P(x) = (x+2)[x[(x-2)c(x)+1]+2]$ $P(3) = (3+2)[3(3-2)c(3)+1]+2]$ $55 = 5[3(c(3)+1)+2]$ $11 = 3c(3)+5$ $c(3) = 2$	$c(x) = 2$	$P(x) = (x+2)[x[(x-2)(2)+1]+2]$

Then, $P(x) = (x+2)[x[(x-2)(2)+1]+2] = (x+2)[x(2x-3)+2] = (x+2)(2x^2-3x+2)$.

Expanding and combining like terms gives the polynomial $P(x) = 2x^3 + x^2 - 4x + 4$.

$P(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 4 = -16 + 4 + 8 + 4 = 0$
$P(0) = 2(0)^3 + (0)^2 - 4(0) + 4 = 4$
$P(2) = 2(2)^3 + (2)^2 - 4(2) + 4 = 16 + 4 - 8 + 4 = 16$
$P(3) = 2(3)^3 + (3)^2 - 4(3) + 4 = 54 + 9 - 12 + 4 = 55$

I can evaluate the function at the given inputs to check my answer.

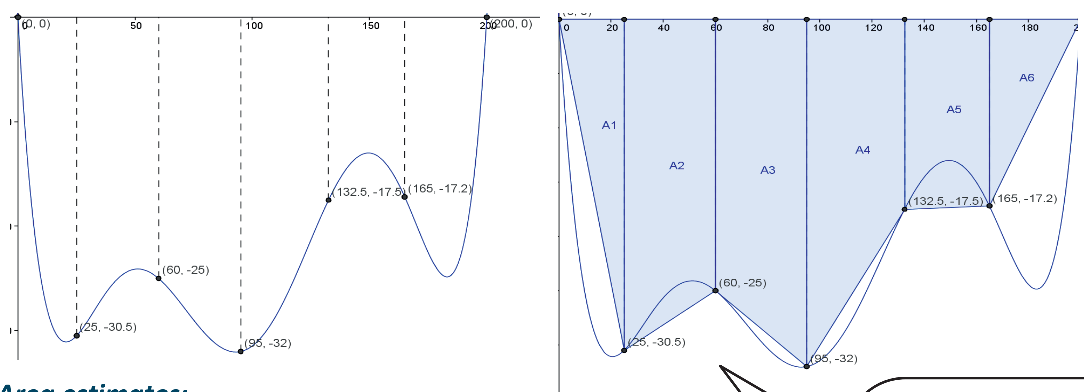
Lesson 21: Modeling Riverbeds with Polynomials

This lesson addresses modeling a cross-section of a riverbed with a polynomial function using technology. Measurements of depth are provided along a riverbed, and these values are used to create a geometric model for the cross-sectional area of the riverbed. This model in turn is used to compute the volumetric flow of the river.

1. Suppose that depths of the riverbed were measured for a cross-section of a different river. Assume the cross-sectional area of the river can be modeled with polynomial Q.

   a. Based on the depths provided, sketch the cross-section of the river, and estimate its area.

$$Q(0) = 0 \qquad Q(25) = -20.5 \qquad Q(60) = -25 \qquad Q(95) = -32$$
$$Q(132.5) = -17 \qquad Q(165) = -22.5 \qquad Q(200) = 0$$



Area estimates:

$$\boldsymbol{A_1 = \frac{1}{2}(25)(30.5) = 381.25}$$
$$\boldsymbol{A_2 = \frac{1}{2}(30.5 + 25)(35) = 971.25}$$
$$\boldsymbol{A_3 = \frac{1}{2}(57)(35) = 997.5}$$
$$\boldsymbol{A_4 = \frac{1}{2}(42.5)(49.5) = 1051.875}$$
$$\boldsymbol{A_5 = \frac{1}{2}(27.5)(34.7) = 477.125}$$
$$\boldsymbol{A_6 = \frac{1}{2}(35)(17.2) = 301}$$

I can estimate the area by dividing the riverbed cross-section into triangular and trapezoidal sections, finding the area of each section, and finding the sum of these areas.

So, the total area can be estimated by

$$\boldsymbol{A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 4180.}$$

The cross-sectional area is approximately $\boldsymbol{4180}$ square feet.

b. Suppose that the speed of the water was measured at $200 \frac{\text{ft}}{\text{min}}$. What is the approximate volumetric flow in this section of the river, measured in gallons per minute?

The volumetric flow is approximately $\mathbf{4180\ ft^2 \left(200 \frac{ft}{min}\right) \approx 836\,000 \frac{ft^3}{min}}$***. Converting to gallons per minute, this is***

$$\mathbf{\left(836\,000 \frac{ft^3}{min}\right)\left(7.48052 \frac{gal}{ft^3}\right) \approx 6\,253\,715 \frac{gal}{min}.}$$

I know $1\ \text{ft}^3$ is equivalent to 7.48052 gallons, and I can use this equivalency to convert the flow rate from $\frac{\text{ft}^3}{\text{min}}$ to gallons per minute.

Lesson 22: Equivalent Rational Expressions

This lesson addresses generating equivalent rational expressions by multiplying or dividing the numerator and denominator by the same factor and noting restrictions for values of variables.

Reduce Rational Expressions to Lowest Terms

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variable that must be excluded to prevent division by zero.

   a. $\frac{3n-12n^2}{18n}$

   $\frac{3n(1-4n)}{3n\cdot 6} = \frac{1-4n}{6}$ ***where*** $n \neq 0$

   I know $n = 0$ is a restricted value because it makes the denominator of the original rational expression zero.

   b. $\frac{ab+bc}{db}$

   $\frac{b(a+c)}{db} = \frac{a+c}{d}$ ***where*** $b \neq 0$ **and** $d \neq 0$.

   I can split the linear term or use the reverse tabular method to factor the trinomial $2n^2 + n - 10$ into the product of two linear factors.

   c. $\frac{2n^2+n-10}{3n-6}$

   $\frac{(2n+5)(n-2)}{3(n-2)} = \frac{2n+5}{3}$ ***where*** $n \neq 2$

   I recognize the numerator as a difference of two squares.

   d. $\frac{(x-y)^2-4a^2}{3y-3x-6a}$

   $\frac{((x-y)-2a)((x-y)+2a)}{-3(x-y+2a)} = \frac{x-y-2a}{-3}$ ***where*** $a \neq \frac{y-x}{2}$

   I can also write this statement as $x \neq y + 2a$ or $y \neq x - 2a$.

2. Write a rational expression with denominator $6b$ that is equivalent to $\frac{2}{3}$.

   $\frac{2}{3}\left(\frac{2b}{2b}\right) = \frac{4b}{6b}$

   I need to multiply the denominator of 3 by $2b$ to make the denominator $6b$.

3. Simplify the following rational expression without using a calculator: $\frac{8^2\cdot 18^3\cdot 10}{12^2\cdot 30}$.

$$\frac{8^2\cdot 18^3\cdot 10}{12^2\cdot 30} = \frac{(2^3)^2\cdot(2\cdot 3^2)^3\cdot 2\cdot 5}{(2^2\cdot 3)^2\cdot 2\cdot 3\cdot 5} = \frac{2^6\cdot 2^3\cdot 3^6\cdot 2\cdot 5}{2^4\cdot 3^2\cdot 2\cdot 3\cdot 5} = \frac{2^{10}\cdot 3^6\cdot 5}{2^5\cdot 3^3\cdot 5} = 2^5\cdot 3^3 = 864$$

Lesson 23: Comparing Rational Expressions

Writing Rational Expressions in Equivalent Forms

1. Rewrite each rational expression as an equivalent rational expression so that all expressions have a common denominator.

$\frac{3}{x^2-2x}, \frac{5}{2x}, \frac{2x+2}{x^2-4}$

$$\frac{3}{x(x-2)}, \frac{5}{2x}, \frac{2x+2}{(x-2)(x+2)}$$

Factoring the denominators will help me determine which factors I need to include in the least common denominator.

The least common denominator is $2x(x-2)(x+2)$.

$$\frac{3(2)(x+2)}{2x(x-2)(x+2)}, \frac{5(x-2)(x+2)}{2x(x-2)(x+2)}, \frac{(2x+2)(2x)}{(x-2)(x+2)(2x)}$$

I need to make sure to multiply the numerator and denominator of a fraction by the same factor(s) so the value stays the same.

Analyze Denominators of Rational Expressions

2. For positive x, determine when the following rational expressions have negative denominators.

   a. $\frac{x+3}{-x^2+8x-18}$

   For any real number x, $-x^2+8x-18$ is always negative.

   $$-x^2+8x-18 = -1(x^2-8x+16)-18+16 = -1(x-4)^2-2,$$

   which is the sum of a nonpositive number and a negative number.

   Since this expression is the product of -1 and a squared number (which cannot be negative), I know this expression cannot be positive.

   b. $\frac{3x^2}{(x-1)(x+2)(x+5)}$

   For positive x, $x+2$ and $x+5$ are always positive. The number $x-1$ is negative when $x<1$, so the denominator is negative when $0<x<1$.

Comparing Rational Expressions

3. If x is a positive number, for which values of x is $x > \frac{1}{x}$?

Since x is a positive number, $x = \frac{x}{x}(x) = \frac{x^2}{x}$.

If $\frac{x^2}{x} > \frac{1}{x}$, then $x^2 > 1$, so $x > 1$.

The inequality is only true if $x > 1$ or $x < -1$, but the problem states x is positive.

4. Determine whether the comparison $\frac{x}{x-1} < \frac{x}{x-5}$ is true for all positive values of x, given that $\frac{6}{6-1} < \frac{6}{6-5}$.

Assuming $x \neq 1$ and $x \neq 5$, $\frac{x}{x-1} = \frac{x(x-5)}{(x-1)(x-5)}$ and $\frac{x}{x-5} = \frac{x(x-1)}{(x-5)(x-1)}$.

If $\frac{x}{x-1} < \frac{x}{x-5}$, then $\frac{x(x-5)}{(x-1)(x-5)} < \frac{x(x-1)}{(x-5)(x-1)}$ and $x^2 - 5x < x^2 - x$ whenever $(x-5)(x-1) > 0$. However, if $(x-5)(x-1) < 0$, then $x^2 - 5x > x^2 - x$, which is only true for $x < 0$. For $1 < x < 5$, $\frac{x}{x-1} > \frac{x}{x-5}$.

I could use a counterexample such as $x = 2$ to show the inequality is not true for every positive value of x.

If the denominator is negative, then multiplying the inequality by the denominator will change the direction of the inequality symbol.

5. A large class has A students, and a smaller class has B students. Use this information to compare the following expressions:

   a. $\frac{A}{A+B}$ and $\frac{A}{B}$

   Since A is positive, the number $A + B > B$, which means $\frac{1}{A+B} < \frac{1}{B}$ and $\frac{A}{A+B} < \frac{A}{B}$.

   The expression $\frac{A}{A+B}$ represents the fraction of the total students in class A, which should be greater than $\frac{1}{2}$ and less than 1. The expression $\frac{A}{B}$ is the ratio of the students in class A to the students in class B, which is greater than 1. Therefore, $\frac{A}{A+B} < \frac{A}{B}$.

   I know this because $A > B$.

b. $\frac{B}{A+B}$ and $\frac{1}{2}$

Since $A > B$, adding B to both sides gives $A + B > B + B$.
It follows that $\frac{1}{A+B} < \frac{1}{B+B}$.

Since $B > 0$, multiplying both sides of the inequality by B does not change the direction of the inequality, so $\frac{B}{A+B} < \frac{B}{B+B}$.

The expression $\frac{B}{A+B}$ represents the fraction of total students in class B.

Since $\frac{B}{B+B} = \frac{B}{2B} = \frac{1}{2}$, it follows that $\frac{B}{A+B} < \frac{1}{2}$. This means that less than half of the total students are in class B.

Lesson 24: Multiplying and Dividing Rational Expressions

Multiply or Divide Rational Expressions

1. Complete the following operation.

   a. $\frac{9}{5}\left(x+\frac{1}{3}\right) \div \frac{3}{10}$

   $$\frac{9}{5}\left(x+\frac{1}{3}\right) \div \frac{3}{10} = \frac{9}{5}\left(x+\frac{1}{3}\right)\cdot\frac{10}{3} = \frac{90}{15}\left(x+\frac{1}{3}\right) = 6\left(x+\frac{1}{3}\right) = 6x+2$$

2. Write each rational expression as an equivalent rational expression in lowest terms.

   a. $\frac{2x^2+9x-5}{3x^2-75}\cdot\frac{3x^2-13x-10}{-6x^2-x+2}$

   $$\frac{(2x-1)(x+5)}{3(x-5)(x+5)}\cdot\frac{(x-5)(3x+2)}{-1(2x-1)(3x+2)} = -\frac{1}{3}$$

   I recognize that $3x^2-75$ is a difference of squares multiplied by the constant 3. I can factor the remaining expressions the way we did earlier in the module.

   b. $\left(\frac{x-1}{x^2+4}\right)^{-2} \div \left(\frac{x^2-2x+1}{x^2+2x-3}\right)$

   $$\left(\frac{x^2+4}{x-1}\right)^2 \div \frac{(x-1)^2}{(x-1)(x+3)} = \frac{(x^2+4)^2}{(x-1)^2}\cdot\frac{(x-1)(x+3)}{(x-1)^2} = \frac{(x^2+4)^2(x+3)}{(x-1)^3}$$

   I remember that division is multiplication by the reciprocal.

   I see the only common factor in both the numerator and denominator of the product is $x-1$.

   c. $\dfrac{\left(\frac{x^2+4x-5}{x^2+2x-3}\right)}{\left(\frac{x^2+3x-10}{x+3}\right)}$

   I know this complex fraction can be written as the quotient of the numerator divided by the denominator.

   $$\frac{x^2+4x-5}{x^2+2x-3} \div \frac{x^2+3x-10}{x+3} = \frac{(x+5)(x-1)}{(x+3)(x-1)}\cdot\frac{x+3}{(x+5)(x-2)} = \frac{1}{x-2}$$

3. Determine which of the following numbers is larger without using a calculator, $\frac{12^{15}}{15^{12}}$ or $\frac{18^{20}}{20^{18}}$.

$$\frac{\mathbf{12^{15}}}{\mathbf{15^{12}}} \div \frac{\mathbf{18^{20}}}{\mathbf{20^{18}}} = \frac{\mathbf{12^{15}}}{\mathbf{15^{12}}} \cdot \frac{\mathbf{20^{18}}}{\mathbf{18^{20}}} = \frac{\left(\mathbf{2^2}\right)^{\mathbf{15}} \cdot \mathbf{3}^{\mathbf{15}} \cdot \left(\mathbf{2^2}\right)^{\mathbf{18}} \cdot \mathbf{5^{18}}}{\mathbf{3^{12}} \cdot \mathbf{5^{12}} \cdot \mathbf{2^{20}} \cdot \left(\mathbf{3^2}\right)^{\mathbf{20}}}$$

$$= \frac{\mathbf{2^{30}} \cdot \mathbf{3^{15}} \cdot \mathbf{2^{36}} \cdot \mathbf{5^{18}}}{\mathbf{3^{12}} \cdot \mathbf{5^{12}} \cdot \mathbf{2^{20}} \cdot \mathbf{3^{40}}} = \frac{\mathbf{2^{46}} \cdot \mathbf{5^6}}{\mathbf{3^{37}}}$$

$$= \left(\frac{\mathbf{10}}{\mathbf{9}}\right)^{\mathbf{6}} \cdot \frac{\mathbf{2^{40}}}{\mathbf{3^{25}}} = \left(\frac{\mathbf{10}}{\mathbf{9}}\right)^{\mathbf{6}} \cdot \left(\frac{\mathbf{2^8}}{\mathbf{3^5}}\right)^{\mathbf{5}} = \left(\frac{\mathbf{10}}{\mathbf{9}}\right)^{\mathbf{6}} \cdot \left(\frac{\mathbf{256}}{\mathbf{243}}\right)^{\mathbf{5}}$$

Since $\frac{10}{9} > 1$***, and*** $\frac{256}{243} > 1$***, the product*** $\left(\frac{10}{9}\right)^6 \cdot \left(\frac{256}{243}\right)^5 > 1$.

This means that $\frac{12^{15}}{15^{12}} > \frac{18^{20}}{20^{18}}$.

I know that if A and B are both positive and $A > B$, then $\frac{A}{B} > 1$. That means if I divide $\frac{\mathbf{12^{15}}}{\mathbf{15^{12}}}$ by $\frac{\mathbf{18^{20}}}{\mathbf{20^{18}}}$ and the quotient is greater than 1, then $\frac{\mathbf{12^{15}}}{\mathbf{15^{12}}}$ is larger than $\frac{\mathbf{18^{20}}}{\mathbf{20^{18}}}$.

I know the product of two fractions whose values are greater than 1 is greater than 1.

4. One of two numbers can be represented by the rational expression $\frac{x}{x-4}$, where $x \neq 0$ and $x \neq 4$.
Find a representation of the second number if the product of the two numbers is 2.

$$\left(\frac{\boldsymbol{x}}{\boldsymbol{x-4}}\right)\boldsymbol{y} = \mathbf{2}$$

$$\frac{\left(\frac{\boldsymbol{x}}{\boldsymbol{x-4}}\right)\boldsymbol{y}}{\left(\frac{\boldsymbol{x}}{\boldsymbol{x-4}}\right)} = \frac{\mathbf{2}}{\left(\frac{\boldsymbol{x}}{\boldsymbol{x-4}}\right)}$$

$$\boldsymbol{y} = \frac{\mathbf{2}}{\left(\frac{\boldsymbol{x}}{\boldsymbol{x-4}}\right)}$$

$$= \mathbf{2} \cdot \frac{\boldsymbol{x-4}}{\boldsymbol{x}}$$

$$= \frac{\boldsymbol{2x-8}}{\boldsymbol{x}}$$

I can define the unknown number as y.

Since $x \neq 0$ and $x \neq 4$, I can divide both sides of the equation by $\frac{x}{x-4}$ without changing their value.

Lesson 25: Adding and Subtracting Rational Expressions

1. Write each sum or difference as a single rational expression.

a. $\frac{\sqrt{5}}{3}+\frac{\sqrt{2}}{5}$

$$\frac{\sqrt{5}}{3}+\frac{\sqrt{2}}{5}=\frac{5\cdot\sqrt{5}}{5\cdot 3}+\frac{3\cdot\sqrt{2}}{3\cdot 5}=\frac{5\sqrt{5}+3\sqrt{2}}{15}$$

This looks almost like the pattern for a square of a difference, but the coefficient of the x^2 term is negative, so I cannot factor the expression.

b. $\frac{x}{x+y}+\frac{y}{y-x}$

$$\frac{x}{x+y}+\frac{y}{y-x}=\frac{x\cdot(y-x)}{(x+y)(y-x)}+\frac{y(x+y)}{(y-x)(x+y)}=\frac{x\cdot(y-x)+y(x+y)}{(x+y)(y-x)}=\frac{y^2+2xy-x^2}{(x+y)(y-x)}$$

c. $\frac{1}{m-n}-\frac{1}{2m+2n}-\frac{m}{n^2-m^2}$

I know $(n-m)=-1\cdot(m-n)$, so I can multiply the third fraction by $\frac{-2}{-2}$ to write an equivalent fraction with the least common denominator of $2(m-n)(m+n)$.

$$\frac{1}{m-n}-\frac{1}{2m+2n}-\frac{m}{n^2-m^2}$$
$$=\frac{1}{m-n}-\frac{1}{2(m+n)}-\frac{m}{(n-m)(n+m)}$$
$$=\frac{2(m+n)}{2(m+n)(m-n)}-\frac{1(m-n)}{2(m+n)(m-n)}+\frac{2m}{2(m-n)(n+m)}$$
$$=\frac{2(m+n)-(m-n)+2m}{2(m-n)(m+n)}=\frac{3m+3n}{2(m-n)(m+n)}$$
$$=\frac{3(m+n)}{2(m-n)(m+n)}$$
$$=\frac{3}{2(m-n)}$$

d. $\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{c+d}{a+b}+\frac{c-d}{a-b}\right)$

$$\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{c+d}{a+b}+\frac{c-d}{a-b}\right)=\left(\frac{a\cdot a}{b\cdot a}-\frac{b\cdot b}{a\cdot b}\right)\left(\frac{(c+d)(a-b)}{(a+b)(a-b)}+\frac{(c-d)(a+b)}{(a-b)(a+b)}\right)$$
$$=\frac{a^2-b^2}{ab}\left(\frac{(c+d)(a-b)+(c-d)(a+b)}{a^2-b^2}\right)$$
$$=\frac{ac-bc+ad-bd+ac+bc-ad-db}{ab}$$
$$=\frac{2ac-2db}{ab}$$

Lesson 26: Solving Rational Equations

1. Solve the following equations, and check for extraneous solutions.

   a. $\frac{6x-18}{x-3} = 6$

   $\frac{6x-18}{x-3} = \frac{6(x-3)}{x-3} = 6$ **if** $x \neq 3$.

   Therefore, the equation is true for all real numbers except 3.

   I know that the denominator of the fraction equals 0 when $x = 3$, which makes the expression undefined for this value.

   b. $\frac{y+1}{y+3} + \frac{1}{y} = \frac{y+9}{y^2+3y}$

   First, we multiply both sides of the equation by the least common denominator $y(y+3)$:

   $y(y+3)\left(\frac{y+1}{y+3}\right) + y(y+3)\left(\frac{1}{y}\right) = y(y+3)\left(\frac{y+9}{y(y+3)}\right)$ ***when*** $y \neq 0$ ***and*** $y \neq -3$

   $$y(y+1) + (y+3) = y+9$$
   $$y^2 + y + y + 3 = y + 9$$
   $$y^2 + y - 6 = 0$$

   I can multiply each side of the equation by the least common denominator $y(y+3)$. I need to remember to exclude values 0 and -3 from the possible solutions.

   The solutions to this equation are -3 ***and*** 2.

   Because -3 ***is an excluded value, the only solution to the original equation is*** $y = 2$.

2. Create and solve a rational equation that has 1 as an extraneous solution.

If 1 is an extraneous solution, there must be a factor of $(x-1)$ in the denominator of one or more of the rational expressions in the equation. Also, 1 must be a potential solution to the rational equation.

For example, 1 is a solution to $x+1=2$, so the rational equation $\frac{x+1}{x-1}=\frac{2}{x-1}$ has an extraneous solution of 1.

Check: Suppose that $\frac{x+1}{x-1}=\frac{2}{x-1}$. Then $\frac{(x+1)(x-1)}{(x-1)}=\frac{2(x-1)}{(x-1)}$ for $x \neq 1$.

Equating numerators gives:

$$\begin{aligned} x^2-1 &= 2x-2 \\ x^2-2x+1 &= 0 \\ (x-1)^2 &= 0 \\ x-1 &= 0 \\ x &= 1 \end{aligned}$$

However, $x=1$ causes division by zero, so 1 is an extraneous solution.

3. Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{7}{8}$? Explain how you know.

If x is an integer, $x+1$ represents the next integer.

$\frac{1}{x}+\frac{1}{x+1}=\frac{7}{8}$

I can model the relationship in the problem with an equation.

$\frac{1(8)(x+1)}{8x(x+1)}+\frac{1(8)(x)}{8x(x+1)}=\frac{7(x)(x+1)}{8x(x+1)}$ ***for $x \neq 0$ and $x \neq -1$***

If I write the expressions with a common denominator, I can equate the numerators for all defined values of x and solve the resulting equation.

Equating numerators gives

$$\begin{aligned} 8(x+1)+8x &= 7x(x+1) \\ 16x+8 &= 7x^2+7x \\ 7x^2-9x-8 &= 0. \end{aligned}$$

Using the quadratic formula gives

$x=\frac{9\pm\sqrt{(-9)^2-4(7)(-8)}}{2(7)}$***, which means that the solutions are $\frac{9+\sqrt{305}}{14}$ and $\frac{9-\sqrt{305}}{14}$. Thus, there are no integer solutions.***

I know that $\sqrt{305}$ is not an integer.

Lesson 27: Word Problems Leading to Rational Equations

1. If two air pumps can fill a bounce house in 10 minutes, and one air pump can fill the bounce house in 30 minutes on its own, how long would the other air pump take to fill the bounce house on its own?

Let x represent the time in minutes it takes the second air pump to fill the bounce house on its own.

$$\frac{1}{30}+\frac{1}{x}=\frac{1}{10}$$

I know the fraction of a bounce house filled by the first air pump in one minute added to the fraction of a bounce house filled by the second air pump in one minute is equal to the fraction of the bounce house filled by both air pumps in one minute.

$$30x\left(\frac{1}{30}+\frac{1}{x}=\frac{1}{10}\right)$$

I know multiplying by $30x$ will not change the solution to the equation because x does not equal zero (it would not make sense for an air pump to take 0 minutes to fill the bounce house).

So, $\frac{30x}{30}+\frac{30x}{x}=\frac{30x}{10}$, and then $x+30=3x$. It follows that $2x=30$, so $x=15$.

It would take the second air pump 15 minutes to fill the bounce house on its own.

2. The difference in the average speed of two cars is 12 miles per hour. The slower car takes 2 hours longer to travel 385 miles than the faster car takes to travel 335 miles. Find the speed of the faster car.

Let t represent the time it takes, in hours, for the faster car to drive 335 miles.

$$\frac{335}{t}-\frac{385}{t+2}=12$$

$$\left(\frac{335(t)(t+2)}{t}\right)-\left(\frac{385(t)(t+2)}{t+2}\right)=12t(t+2)$$

I know distance is the average speed multiplied by time, so average speed is distance divided by time.

I also know that the difference in the average speeds of the cars is 12 miles per hour.

Thus, $335(t+2)-385t=12t^2+24t$ and $335t+670-385t=12t^2+24t$. Combining like terms gives $12t^2+74t-670=0$, so $2(t-5)(6t+67)=0$.

The only possible solution is 5.

Solving $(6t+67)=0$ produces a negative solution for t, which does not make sense in this context.

Because $\frac{335}{5}=67$, the speed of the faster car is 67 miles per hour.

Average speed is distance divided by time.

3. Consider an ecosystem of squirrels on a college campus that starts with 30 squirrels and can sustain up to 150 squirrels. An equation that roughly models this scenario is

$$P = \frac{150}{1 + \frac{4}{t+1}},$$

where P represents the squirrel population in year t of the study.

a. Solve this equation for t. Describe what the resulting equation represents in the context of this problem.

Since $P = \frac{150}{1+\frac{4}{t+1}}$, it follows that $P\left(1 + \frac{4}{t+1}\right) = 150$, and then $1 + \frac{4}{t+1} = \frac{150}{P}$.

I can multiply this equation by the least common denominator $P(t+1)$ because I know neither factor has a value of zero.

Then

$$\begin{aligned} P(t+1)\left(1 + \frac{4}{t+1}\right) &= P(t+1)\left(\frac{150}{P}\right) \\ P(t+1) + 4P &= 150(t+1) \\ Pt + P + 4P &= 150t + 150 \\ 150t - Pt &= 5P - 150 \\ t(150 - P) &= 5P - 150 \end{aligned}$$

$$t = \frac{5P - 150}{150 - P}$$

I need to assume that $P \neq 150$ so that I can divide both sides of the equation by $150 - P$.

The resulting equation represents the relationship between the number of squirrels, P, and the year, t. If we know how many squirrels we have, between 30 and 150, we will know how long it took for the squirrel population to grow from 30 to that value, P. If the population is 30, then this equation says we are in year 0 of the study, which fits with the given scenario.

b. At what time does the population reach 100 squirrels?

When $P = 100$, then $t = \frac{5(100)-150}{150-100} = \frac{350}{50} = 7$; therefore, the squirrel population is 100 in year 7 of the study.

Lesson 28: A Focus on Square Roots

Rationalize the Denominator

I know that when the denominator of a rational expression is in the form $a + \sqrt{b}$, I can rationalize the denominator by multiplying both the numerator and denominator by $a - \sqrt{b}$.

1. Rationalize the denominator in the following expressions:

a. $\frac{3x}{5-\sqrt{x}}$

If $x \neq 25$, then

$$\frac{3x}{5-\sqrt{x}} = \frac{3x}{5-\sqrt{x}} \cdot \frac{5+\sqrt{x}}{5+\sqrt{x}} = \frac{3x(5)+3x(\sqrt{x})}{5(5)+5\sqrt{x}-5\sqrt{x}-(\sqrt{x})^2} = \frac{15x+3x\sqrt{x}}{25-x}.$$

b. $\frac{3}{\sqrt{x+4}}$

If $x > -4$, then

$$\frac{3}{\sqrt{x+4}} = \frac{3}{\sqrt{x+4}} \cdot \frac{\sqrt{x+4}}{\sqrt{x+4}} = \frac{3\sqrt{x+4}}{x+4}.$$

I know that $\sqrt{a} \cdot \sqrt{a} = a$ for non-negative values of a. In this case, $a = x + 4$.

Solve Equations That Contain Radicals

2. Solve each equation, and check the solutions.

a. $3\sqrt{x+3} + 4 = -2$

$$3\sqrt{x+3} = -6$$
$$\sqrt{x+3} = -2$$

I can see that there are no real solutions because the principal square root of a number cannot be negative.

$$\left(\sqrt{x+3}\right)^2 = (-2)^2$$
$$x + 3 = 4$$
$$x = 1$$

Check: If $x = 1$ is a solution, then $\sqrt{1+3} = -2$, but $2 \neq -2$, so 1 is an extraneous solution.

There are no real solutions to the original equation.

b. $\sqrt{x+5} - 2\sqrt{x-1} = 0$

I first need to separate the square root expressions on opposite sides of the equal sign and then square both sides of the equation.

$$\sqrt{x+5} = 2\sqrt{x-1}$$
$$\left(\sqrt{x+5}\right)^2 = \left(2\sqrt{x-1}\right)^2$$
$$x+5 = 4(x-1)$$
$$x+5 = 4x-4$$
$$3x = 9$$
$$x = 3$$

Check: $\sqrt{3+5} - 2\sqrt{3-1} = \sqrt{8} - 2\sqrt{2} = 2\sqrt{2} - 2\sqrt{2} = 0$, so 3 is a valid solution.

c. $\sqrt{x^2 - 5x} = 6$

$$\left(\sqrt{x^2 - 5x}\right)^2 = 6^2$$
$$x^2 - 5x = 36$$
$$x^2 - 5x - 36 = 0$$
$$(x - 9)(x + 4) = 0$$

Either $x = 9$ or $x = -4$

Check: $\sqrt{9^2 - 5(9)} = \sqrt{81 - 45} = \sqrt{36} = 6;$
$\sqrt{(-4)^2 - 5(-4)} = \sqrt{16 + 20} = \sqrt{36} = 6$

Both 9 and -4 are solutions to the original equation.

d. $\frac{3}{\sqrt{x}-1} = 2$

$$\frac{3}{\sqrt{x}-1} \cdot (\sqrt{x} - 1) = 2(\sqrt{x} - 1) \text{ for } x \neq 1$$
$$3 = 2\sqrt{x} - 2(1)$$
$$5 = 2\sqrt{x}$$
$$\frac{5}{2} = \sqrt{x}$$
$$\frac{25}{4} = x$$

I could also solve this equation by rationalizing the denominator of $\frac{3}{\sqrt{x}-1}$, but multiplying by the least common denominator takes fewer steps.

Check: $\frac{3}{\sqrt{\frac{25}{4}}-1} = \frac{3}{\frac{5}{2}-1} = \frac{3}{\left(\frac{3}{2}\right)} = 3 \cdot \frac{2}{3} = 2$

The only solution is $\frac{25}{4}$.

e. $\sqrt[3]{2x + 1} + 5 = 2$

$$\sqrt[3]{2x + 1} = -3$$
$$\left(\sqrt[3]{2x + 1}\right)^3 = (-3)^3$$
$$2x + 1 = -27$$
$$2x = -28$$
$$x = -14$$

Check: $\sqrt[3]{2(-14) + 1} + 5 = \sqrt[3]{-27} + 5 = -3 + 5 = 2$

The only solution is -14.

Lesson 29: Solving Radical Equations

Solve each equation.

1. $\sqrt{3x+1} = 1 - 2x$

I know that squaring an equation does not always generate an equivalent equation, so I will need to check for extraneous solutions possibly introduced by squaring.

$$\left(\sqrt{3x+1}\right)^2 = (1-2x)^2$$
$$3x+1 = 1^2 - 2(1)(2x) + (2x)^2$$
$$3x+1 = 1 - 4x + 4x^2$$
$$4x^2 - 7x = 0$$
$$x(4x-7) = 0$$
$$x = 0 \text{ or } x = \frac{7}{4}$$

If** $x = 0$**, then** $\sqrt{3x+1} = \sqrt{3(0)+1} = 1$**, and** $1 - 2x = 1 - 0 = 1$**, so** 0 **is a valid solution.

If** $x = \frac{7}{4}$**, then** $\sqrt{3x+1} = \sqrt{3\left(\frac{7}{4}\right)+1} = \sqrt{\frac{25}{4}} = \frac{5}{2}$**, and** $1 - 2x = 1 - \frac{14}{4} = -\frac{5}{2}$**, so** $\frac{7}{4}$ **is not a solution.

The only solution is** 0**.

2. $\sqrt{x-5} + \sqrt{2x-3} = 4$

I can use the pattern $(a-b)^2 = a^2 - 2ab + b^2$ to expand the expression $\left(4 - \sqrt{2x-3}\right)^2$. I will be left with a term containing a square root, which I will need to isolate.

$$\sqrt{x-5} = 4 - \sqrt{2x-3}$$
$$\left(\sqrt{x-5}\right)^2 = \left(4 - \sqrt{2x-3}\right)^2$$
$$x - 5 = 4^2 - 2(4)\sqrt{2x-3} + \left(\sqrt{2x-3}\right)^2$$
$$x - 5 = 16 - 8\sqrt{2x-3} + 2x - 3$$
$$x - 5 = 2x + 13 - 8\sqrt{2x-3}$$
$$x + 18 = 8\sqrt{2x-3}$$
$$(x+18)^2 = \left(8\sqrt{2x-3}\right)^2$$
$$x^2 + 36x + 324 = 64(2x-3)$$
$$x^2 + 36x + 324 = 128x - 192$$
$$x^2 - 92x + 516 = 0$$
$$(x-6)(x-86) = 0$$
$$x = 6 \text{ or } x = 86$$

Now that I have isolated the radical term, I can square both sides of the equation, which will create a quadratic equation I can solve.

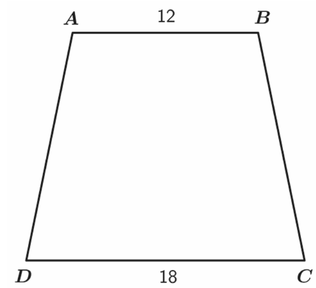
If** $x = 6$**, then** $\sqrt{x-5} + \sqrt{2x-3} = \sqrt{6-5} + \sqrt{2(6)-3} = \sqrt{1} + \sqrt{9} = 4$**, so** 6 **is a solution.

If** $x = 86$**, then** $\sqrt{x-5} + \sqrt{2x-3} = \sqrt{86-5} + \sqrt{2(86)-3} = \sqrt{81} + \sqrt{169} = 22 \neq 4$**, so** 86 **is not a solution to the original equation.

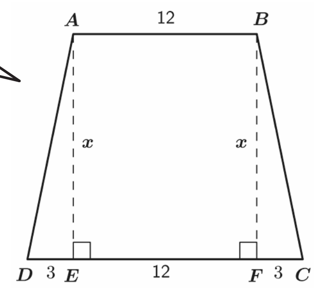
The only solution to the original equation is** 6**.

3. Consider the trapezoid $ABCD$ shown to the right, where $AD = BC$.

   a. If the length of $\overline{AB}$ is 12, the length of $\overline{DC}$ is 18, and the height of the trapezoid is x, write an expression for the perimeter of trapezoid $ABCD$ in terms of x.



Since $ABCD$ is an isosceles trapezoid, I can decompose it into a rectangle of width 12 and height x, and two congruent right triangles.



$$(AD)^2 = (DE)^2 + (AE)^2$$
$$(AD)^2 = 3^2 + x^2$$
$$AD = \sqrt{9 + x^2}$$

I can use the Pythagorean theorem to find the length AD, which has the same value as BC.

Since $AB + BC + CD + AD = 12 + \sqrt{9 + x^2} + 18 + \sqrt{9 + x^2}$, the perimeter of $ABCD$ is $30 + 2\sqrt{9 + x^2}$.

   b. Find the value of x for which the perimeter of trapezoid $ABCD$ is equal to 60 cm.

$$30 + 2\sqrt{9 + x^2} = 60$$
$$2\sqrt{9 + x^2} = 30$$
$$\sqrt{9 + x^2} = 15$$
$$9 + x^2 = 15^2$$
$$9 + x^2 = 225$$
$$x^2 = 216$$
$$x = \sqrt{216} = 6\sqrt{6}$$

The height of trapezoid $ABCD$ is $6\sqrt{6}$ centimeters when the perimeter is 60 centimeters.

Lesson 30: Linear Systems in Three Variables

1. Solve the following systems of equations:

a. $r = 3(s + t)$
$t = s - r$
$r = 2s - 1$

If I substitute the expression $(2s - 1)$ for r into the first and second equations, I can create two equations with two variables. Since the coefficients of s are opposites in the resulting equations, I can then add the equations to eliminate s.

$$\begin{aligned} 2s - 1 &= 3(s + t) \\ t &= s - (2s - 1) \end{aligned} \qquad \begin{aligned} -s - 3t &= 1 \\ s + t &= 1 \end{aligned}$$

Adding the equations gives $-2t = 2$, so $t = -1$.

Then, $s + t = 1$ so $s + (-1) = 1$ and then $s = 2$.
Since $r = 2s - 1$, it follows that $r = 2(2) - 1$ and then $r = 3$.

Solution: $(3, 2, -1)$

Once I find the value of t, I can back substitute its value into $s + t = 1$ to find the value of s and then back substitute the value of s into $r = 2s - 1$ to find the value of r.

b. $p + q + 3r = 0$
$5p - 2q + 3r = 7$
$p - q + r = 2$

$$\begin{array}{r} p + q + 3r = 0 \\ \underline{+\ p - q + \ r = 2} \\ 2p \qquad + 4r = 2 \end{array} \qquad \begin{array}{r} 2p + 2q + 6r = 0 \\ \underline{+\ 5p - 2q + 3r = 7} \\ 7p \qquad + 9r = 7 \end{array}$$

If I add the first and third equations, I can eliminate q. I can also eliminate q by adding twice the first equation to the second equation.

Multiply both sides of $2p + 4r = 2$ by -7, and multiply both sides of $7p + 9r = 7$ by 2.

$$\begin{aligned} -7(2p + 4r) &= -7(2) \\ 2(7p + 9r) &= 2(7) \end{aligned} \qquad \begin{aligned} -14p - 28r &= -14 \\ 14p + 18r &= 14 \end{aligned}$$

I can add these equations to solve for r.

$$\begin{array}{r} -14p - 28r = -14 \\ \underline{+\ 14p + 18r = 14} \\ -10r = 0 \end{array}$$

Then, $r = 0$.
Since $7p + 9r = 7$ and $r = 0$, it follows that $7p + 0 = 7$ and $p = 1$.
Then since $p + q + 3r = 0$, it follows that $1 + q + 3(0) = 0$ and then $q = -1$.

Solution: $(1, -1, 0)$

c. $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=0$

$\frac{3}{x}-\frac{1}{z}=0$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6$

If I define variables to represent $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, I can rewrite the system of equations in a format similar to that of the other systems I have solved.

Let $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$. Then the system becomes

$$\begin{aligned} a+b-c&=0\\ 3a\quad\;\; -c&=0\\ a+b+c&=6 \end{aligned}$$

I can add pairs of equations to eliminate c.

$$\begin{array}{r} a+b-c=0\\ \underline{+\;a+b+c=6}\\ 2a+2b\quad\;=6 \end{array}\qquad\qquad \begin{array}{r} a+b+c=6\\ \underline{+\;3a\quad\;\;-c=0}\\ 4a+b\quad\;\;=6 \end{array}$$

$$\begin{array}{r} 4a+4b=\;\;12\\ \underline{-\;4a-\;\;b=-6}\\ 3b=\;\;\;6 \end{array}$$

I can combine the new equations to eliminate a.

So, $b=2$. Since $4a+b=6$ and $b=2$, we see that $a=1$.

Then $3a-c=0$ and $a=1$ gives $c=3$.

Since $a=\frac{1}{x}$ and $a=1$, we know $x=1$.

Since $b=\frac{1}{y}$ and $b=2$, we know $y=\frac{1}{2}$.

Since $c=\frac{1}{z}$ and $c=3$, we know $z=\frac{1}{3}$.

I need to be sure to write the solution to the original system by finding values of x, y, and z.

Solution: $\left(1,\frac{1}{2},\frac{1}{3}\right)$

2. A chemist needs to make 50 ml of a 12% acid solution. If he uses 10% and 20% solutions to create the 12% solution, how many ml of each does he need?

Let x represent the amount of 10% solution used and y represent the amount of 20% solution used.

$$x + y = 50$$
$$0.1x + 0.2y = 0.12(50)$$

The sum of the amount of the two solutions used is 50 ml.

The amount of acid in a solution is the product of the volume of the solution and its concentration.

Multiply the second equation by 10: $10(0.1x + 0.2y) = 10(0.12)(50)$ becomes $x + 2y = 60$.

$$\begin{aligned} x + 2y &= 60 \\ \underline{-x - y} &= \underline{-50} \\ y &= 10 \end{aligned}$$

I can add -1 times the first equation to ten times the second equation to find the value of y .

Since $y = 10$ and $x + y = 50$, we have $x = 40$. The chemist used 40 ml of the 10% solution and 10 ml of the 20% solution.

Lesson 31: Systems of Equations

1. Where do the lines given by $y = x - b$ and $y = 3x + 1$ intersect?

$$\begin{aligned} x - b &= 3x + 1 \\ 2x &= -b - 1 \\ x &= \frac{-b-1}{2} \\ y &= x - b \\ y &= \frac{-b-1}{2} - b = \frac{-b-1-2b}{2} = \frac{-3b-1}{2} \end{aligned}$$

I know that, at the point of intersection, the y-coordinates will be the same, so I can set $x - b = 3x + 1$.

The point of intersection for the lines is $\left(\frac{-b-1}{2}, \frac{-3b-1}{2}\right)$.

2. Find all solutions to the following systems of equations. Illustrate each system with a graph.

   a. $(x-1)^2 + (y+2)^2 = 9$
   $2x - y = 7$

   First, $2x - y = 7$***, so*** $y = 2x - 7$.

   I can solve the linear equation for y and substitute the result into the quadratic equation.

   Then $(x-1)^2 + \left((2x-7)+2\right)^2 = 9$***, so*** $(x-1)^2 + (2x-5)^2 = 9$.

   $$\begin{aligned} x^2 - 2x + 1 + 4x^2 - 20x + 25 &= 9 \\ 5x^2 - 22x + 17 &= 0 \\ (5x-17)(x-1) &= 0 \end{aligned}$$

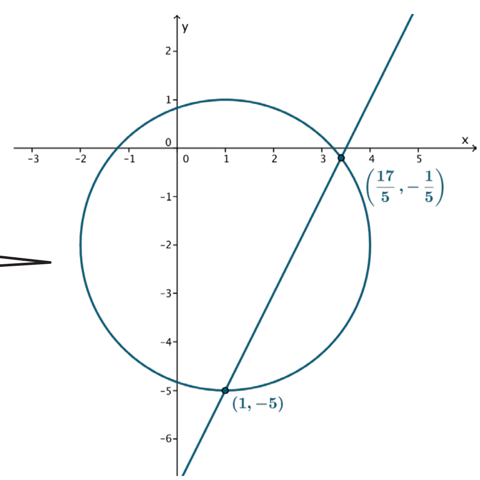
   I should back substitute these values of x into the linear equation $2x - y = 7$ to avoid extraneous solutions that might arise if I used the more complicated equation.

   Then $x = \frac{17}{5}$ ***or*** $x = 1$.

   If $x = \frac{17}{5}$, $y = 2\left(\frac{17}{5}\right) - 7 = -\frac{1}{5}$.

   If $x = 1$, $y = 2(1) - 7 = -5$.

   The points of intersection are $\left(\frac{17}{5}, -\frac{1}{5}\right)$ ***and*** $(1, -5)$.



My graph shows the two points of intersection.

b. $y = 4x - 11$
$y = x^2 - 4x + 5$

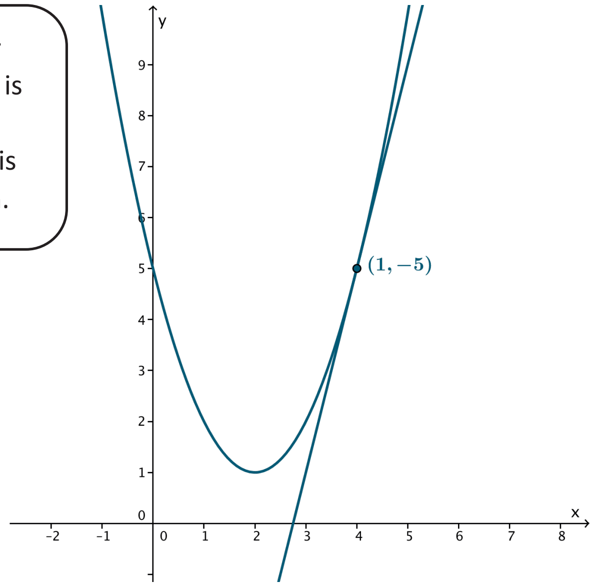
$\boldsymbol{4x - 11 = x^2 - 4x + 5}$
$\boldsymbol{x^2 - 8x + 16 = 0}$
$\boldsymbol{(x - 4)(x - 4) = 0}$

The only solution is 4.

If $x = 4$, then $y = 4(4) - 11 = 5$.

The only point of intersection is $(4, 5)$.

The repeated factor indicates that there is one point of intersection, which is shown on the graph.



3. Find all values of k so that the following system has no solutions.
$(y - k)^2 = x + 3$
$y = -2x + k$

$$\boldsymbol{\left((-2x + k) - k\right)^2 = x + 3}$$
$$\boldsymbol{(-2x)^2 = x + 3}$$
$$\boldsymbol{4x^2 - x - 3 = 0}$$
$$\boldsymbol{(4x + 3)(x - 1) = 0}$$
$$\boldsymbol{x = 1 \textit{ or } x = -\frac{3}{4}}$$

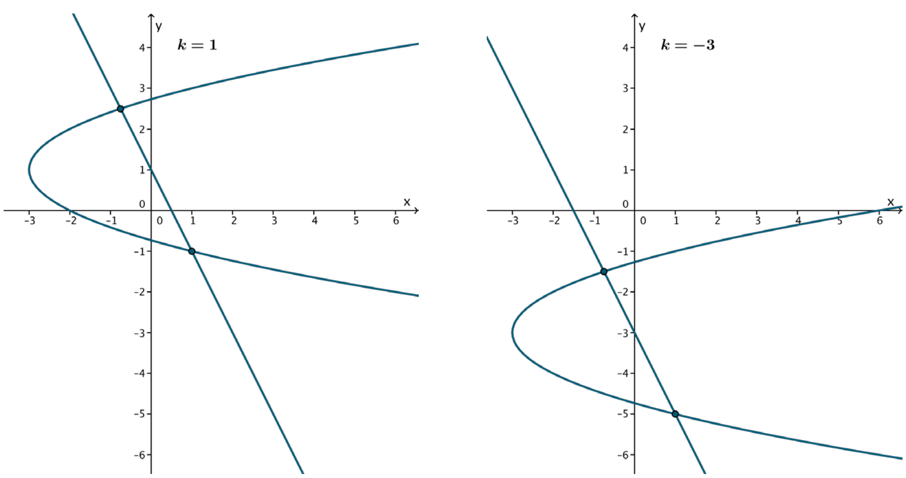
Substitute $-2x + k$ for y in the first equation, and solve for x.

If $x = 1$, then $y = -2(1) + k = k - 2$.

If $x = -\frac{3}{4}$, then $y = -2\left(-\frac{3}{4}\right) + k = k + \frac{3}{2}$, so the intersection points are $(1, k - 2)$ and $\left(-\frac{3}{4}, k + \frac{3}{2}\right)$.

The graphs will intersect in two locations regardless of the value of k. There are no values for k so that the system has no solutions.

These are examples of the graphs of systems for two different values of k.



Lesson 32: Graphing Systems of Equations

1. Use the distance formula to find the length of the diagonal of a rectangle whose vertices are $(1,3)$, $(4,3)$, $(4,9)$, and $(1,9)$.

Using the vertices $(4, 3)$ and $(1, 9)$, we have $d = \sqrt{(1-4)^2 + (9-3)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$.

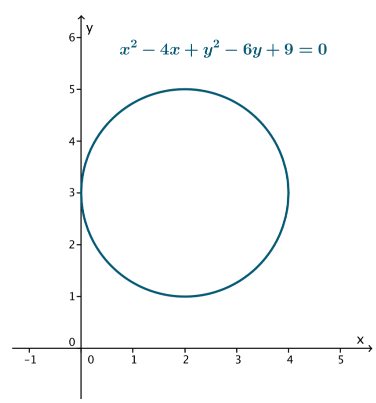
Since the diagonals of a rectangle are congruent, I can use any two nonadjacent vertices to find the length of the diagonal.

I know $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ by the distance formula.

The diagonal is $3\sqrt{5}$ units long.

2. Write an equation for the circle with center $(2, 3)$ in the form $(x-h)^2 + (y-k)^2 = r^2$ that is tangent to the y-axis, where the center is (h, k) and the radius is r. Then write the equation in the standard form $x^2 + ax + y^2 + by + c = 0$, and construct the graph of the equation.

Since the center of the circle has coordinates $(2, 3)$ and the circle is tangent at to the y-axis, the circle must pass through the point $(0, 3)$, which means the radius is 2 units long.



If the graph of the circle does not contain the point $(0, 3)$, either the circle does not intersect the y-axis, or it intersects the y-axis in two places.

Equation for circle: $(x-2)^2 + (y-3)^2 = 2^2$

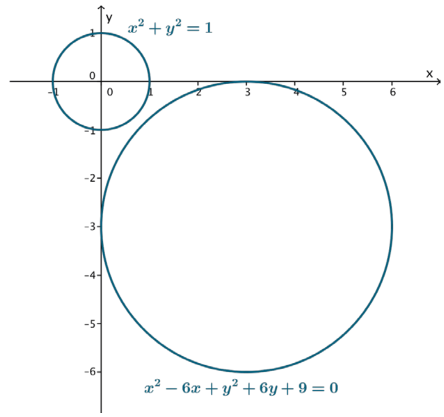
$$(x-2)^2 + (y-3)^2 = 2^2$$
$$x^2 - 4x + 4 + y^2 - 6x + 9 = 4$$

Standard form: $x^2 - 4x + y^2 - 6y + 9 = 0$

3. By finding the radius of each circle and the distance between their centers, show that the circles with equations $x^2 + y^2 = 1$ and $x^2 - 6x + y^2 + 6y + 9 = 0$ do not intersect. Illustrate graphically.

The graph of $x^2 + y^2 = 1$ has center $(0, 0)$ and radius 1 unit.

$$\begin{aligned} x^2 - 6x + y^2 + 6y + 9 &= 0 \\ \left(x^2 - 6x + 9\right) + \left(y^2 + 6y + 9\right) &= 9 \\ (x-3)^2 + (y+3)^2 &= 9 \end{aligned}$$



The graph of $x^2 - 6x + y^2 + 6y + 9 = 0$ is a circle with center $(3, -3)$ and radius 3 units.

Since $\sqrt{(3-0)^2 + (-3-0)^2} = \sqrt{18}$, the distance between the centers of the circles is $\sqrt{18}$ units.

I can use the distance formula to find the distance between the centers $(0, 0)$ and $(3, -3)$.

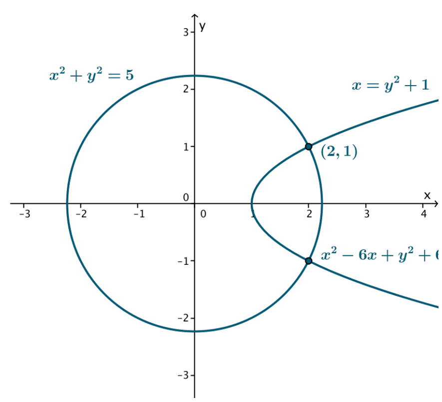
Since $3 + 1 = 4$, the sum of the radii is 4 units.
Since $\sqrt{18} > 4$, the circles do not overlap.

I know that if two circles intersect, the distance between the centers is less than or equal to the sum of their radii. Since the distance between the centers is greater than the sum of the radii, the circles cannot overlap.

4. Solve the system $x = y^2 + 1$ and $x^2 + y^2 = 5$.
Illustrate graphically.

I can substitute $(x-1)$ for y^2 in the second equation.

$$\begin{aligned} x^2 + y^2 &= 5 \\ x^2 + (x-1) &= 5 \\ x^2 + x - 1 - 5 &= 0 \\ x^2 + x - 6 &= 0 \\ (x-2)(x+3) &= 0 \\ x = 2 \text{ or } x &= -3 \end{aligned}$$



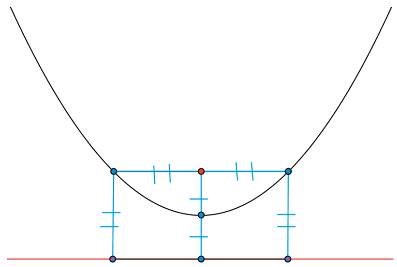
Because $x = y^2 + 1$, it follows that $y^2 = x - 1$.

If $x = 2$, then $y^2 = 2 - 1$, so $y^2 = 1$. Thus either $y = 1$ or $y = -1$.
If $x = -3$, then $y^2 = -3 - 1$, so $y^2 = -4$ which has no real solution.

The points of intersection are $(2, 1)$ and $(2, -1)$.

Lesson 33: The Definition of a Parabola

1. Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given the parabola, its focus, and directrix. Measure the segments that you drew in either inches or centimeters to confirm the accuracy of your sketches.



2. Find the values of x for which the point $(x, 2)$ is equidistant from $(3,5)$ and the line $y = -3$.

Because $\mathbf{2 - (-3) = 5}$***, the distance from*** $\mathbf{(x, 2)}$ ***to the line*** $\mathbf{y = -3}$ ***is*** $\mathbf{5}$ ***units.***

I know the distance between a point (a, b) and a horizontal line $y = k$ is $|k - b|$.

The distance from $\mathbf{(x, 2)}$ ***to*** $\mathbf{(3, 5)}$ ***is*** $\mathbf{\sqrt{(x-3)^2 + (2-5)^2} = \sqrt{x^2 - 6x + 9 + 9} = \sqrt{x^2 - 6x + 18}}$.

Then,

$$\begin{aligned} \sqrt{x^2 - 6x + 18} &= 5 \\ x^2 - 6x + 18 &= 25 \\ x^2 - 6x - 7 &= 0 \\ (x-7)(x+1) &= 0 \\ x = 7 \text{ or } x &= -1 \end{aligned}$$

I can set the distances I calculated equal to each other.

The points $\mathbf{(7, 2)}$ ***and*** $\mathbf{(-1, 2)}$ ***are both equidistant from the point*** $\mathbf{(3, 5)}$ ***and the line*** $\mathbf{y = -3}$.

3. Consider the equation $y = \frac{1}{4}x^2 + 2x$.

   a. Find the coordinates of the points on the graph of $y = \frac{1}{4}x^2 + 2x$ whose x-values are 0 and 2.

   If $\mathbf{x = 0}$***, then*** $\mathbf{y = \frac{1}{4}(0)^2 + 2(0) = 0}$.

   If $\mathbf{x = 2}$***, then*** $\mathbf{y = \frac{1}{4}(2)^2 + 2(2) = 5}$.

   The coordinates are $\mathbf{(0, 0)}$ ***and*** $\mathbf{(2, 5)}$.

b. Show that each point in part (a) is equidistant from the point $(-4,-3)$ and the line $y=-5$.

The distance from the point $(0,0)$ to the line $y=-5$ is 5.

The distance between $(0,0)$ and $(-4,-3)$ is $\sqrt{(-4-0)^2+(-3-0)^2}=\sqrt{25}=5$.

Thus, the point $(0,0)$ is equidistant from the point $(-4,-3)$ and the line $y=-5$.

The distance from the point $(2,5)$ to the line $y=-5$ is 10.

The distance between $(2,5)$ and $(-4,-3)$ is $\sqrt{(-4-2)^2+(-3-5)^2}=\sqrt{100}=10$.

Thus, the point $(2,5)$ is equidistant from the point $(-4,-3)$ and the line $y=-5$.

c. Show that if the point with coordinates (x,y) is equidistant from the point $(-4,-3)$ and the line $y=-5$, then $y=\frac{1}{4}x^2+2x$.

The distance between the point with coordinates (x,y) and the line $y=-5$ is $y+5$.

The distance between the points (x,y) and $(-4,-3)$ is $\sqrt{(x+4)^2+(y+3)^2}$.

So, if $y+5=\sqrt{(x+4)^2+(y+3)^2}$, then

$$\begin{aligned}(y+5)^2&=(x+4)^2+(y+3)^2\\ y^2+10y+25&=x^2+8x+16+y^2+6y+9\\ 10y-6y+25&=x^2+8x+16+y^2-y^2+9\\ 4y&=x^2+8x\\ y&=\frac{1}{4}(x^2+8x)\\ y&=\frac{1}{4}x^2+2x\end{aligned}$$

Since the point (x,y) is equidistant from $(-4,-3)$ and the line, I set these distances equal to find the relationship between y and x.

4. Derive the analytic equation of a parabola with focus $(2,6)$ and directrix on the x-axis.

The distance between the point with coordinates (x,y) and the line $y=0$ is y, and the distance between the points (x,y) and $(2,6)$ is $\sqrt{(x-2)^2+(y-6)^2}$. These distances are equal.

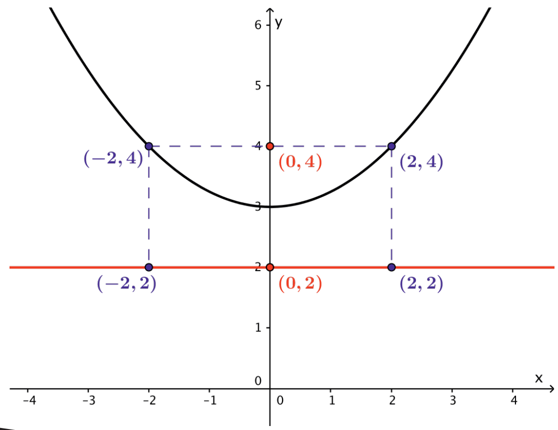
$$\begin{aligned}y&=\sqrt{(x-2)^2+(y-6)^2}\\ y^2&=(x-2)^2+y^2-12y+36\\ y^2-(y^2-12y)&=(x-2)^2+36\\ 12y&=(x-2)^2+36\\ y&=\frac{1}{12}((x-2)^2+36)\\ y&=\frac{1}{12}(x-2)^2+3\end{aligned}$$

I know the equation of the parabola will have the form $y=a(x-h)^2+k$, so I am leaving the expression $(x-2)^2$ in factored form and expanding the expression $(y-6)^2$.

Lesson 34: Are All Parabolas Congruent?

1. Use the definition of a parabola to sketch the parabola defined by the given focus and directrix. Then write an analytic equation for each parabola.

   a. Focus: $(0,4)$ Directrix: $y = 2$



The vertex lies along the y-axis, halfway between the focus and directrix.

The vertex of the parabola is $(0, 3)$.

Since the distance from the vertex to the focus is 1 unit, $\frac{1}{2}p = 1$, and then $p = 2$.

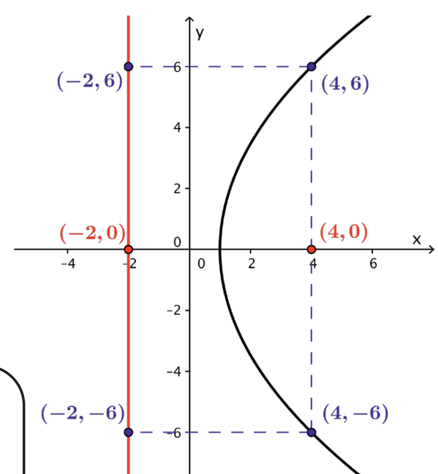
I know the distance from the vertex to the focus is $\frac{1}{2}p$.

The parabola opens up, so $y = \frac{1}{2p}(x - h)^2 + k$.

Then $y = \frac{1}{4}x^2 + 3$.

I know the parabola opens upward because the directrix is beneath it. Therefore, I can use the general form for the equation of a parabola that opens up with vertex (h, k).

   b. Focus: $(4,0)$ Directrix: $x = -2$



The vertex of the parabola is $(1, 0)$.

Since the distance from the vertex to the focus is 3 units, $\frac{1}{2}p = 3$, and then $p = 6$.

The parabola opens to the right, so $x = \frac{1}{2p}(y - k)^2 + h$, which gives $x = \frac{1}{12}y^2 + 1$.

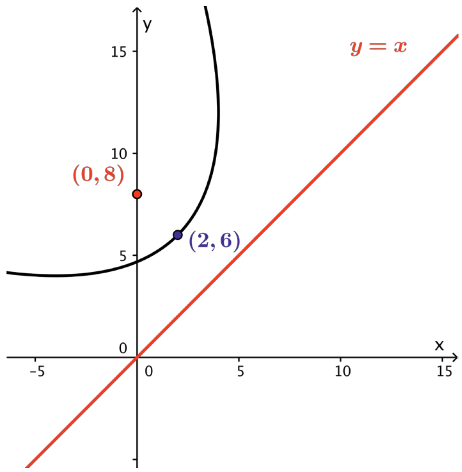
I know the parabola opens to the right because the directrix is to the left of it. Because of this, I can use the general form for the equation of a parabola that opens to the right with vertex (h, k).

c. Explain how you can tell whether the parabolas in parts (a) and (b) are congruent.

The parabolas are not congruent because they do not have the same value of p.

2. Let P be the parabola with focus $(0,8)$ and directrix $y = x$.

a. Sketch this parabola. Then write an equation in the form $y = \frac{1}{2a}x^2$ whose graph is a parabola that is congruent to P.



The vertex of the parabola is equidistant from the focus and directrix along the line through the focus that is perpendicular to the directrix. The equation of that line is $y - 8 = -1(x - 0)$, which simplifies to $y = -x + 8$.

I know this line will have slope -1 because that is the slope of a line perpendicular to a line of slope 1, such as $y = x$.

The point where the line $y = -x + 8$ intersects the directrix has coordinates (x, x), so $x = -x + 8$.

Then $2x = 8$, so $x = 4$, and the intersection point on the directrix is $(4, 4)$.

The vertex is the point halfway between $(0, 8)$ and $(4, 4)$, which is $\left(\frac{0+4}{2}, \frac{8+4}{2}\right) = (2, 6)$.

The distance from the vertex to the focus is $\sqrt{(0-2)^2 + (8-6)^2} = \sqrt{4+4} = 2\sqrt{2}$.

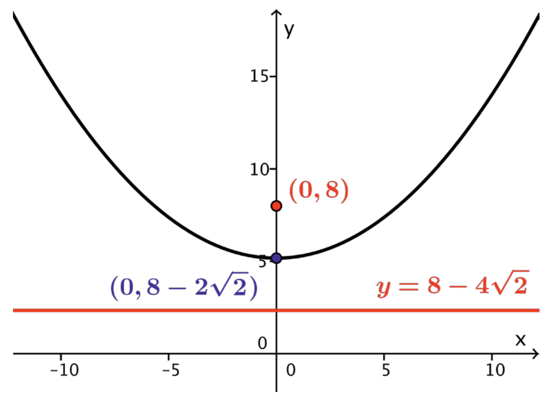
Then, $2\sqrt{2} = \frac{1}{2}p$, so $p = 4\sqrt{2}$. It follows that $y = \frac{1}{8\sqrt{2}}x^2$.

b. Write an equation whose graph is P', the parabola congruent to P that results after P is rotated clockwise $45°$ about the focus.

The equation for the directrix is $y = 8 - 4\sqrt{2}$.
The equation for the parabola is $y = \frac{1}{8\sqrt{2}}x^2 + 8 - 2\sqrt{2}$.

I know the directrix will be a horizontal line, and the distance between the focus, whose coordinates are $(0, 8)$ and the directrix along the y-axis, is $4\sqrt{2}$.

The distance from the vertex to the focus is $2\sqrt{2}$.



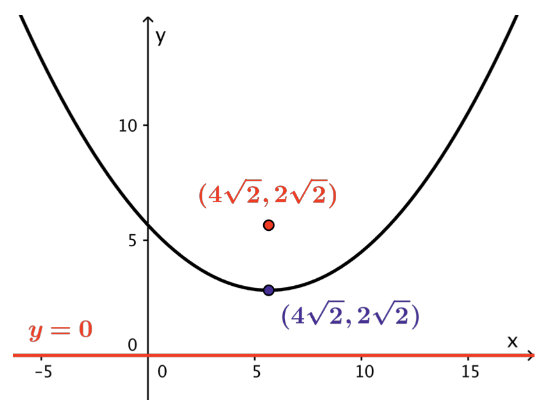
c. Write an equation whose graph is P'', the parabola congruent to P that results after the directrix of P is rotated $45°$ clockwise about the origin.

The focus is $(4\sqrt{2}, 4\sqrt{2})$, and the directrix is the x-axis.

The distance between the origin and the point on the directrix that aligns with the focus is $4\sqrt{2}$, so the directrix, vertex, and focus will align at $x = 4\sqrt{2}$. The distance from the directrix to the focus is $4\sqrt{2}$.

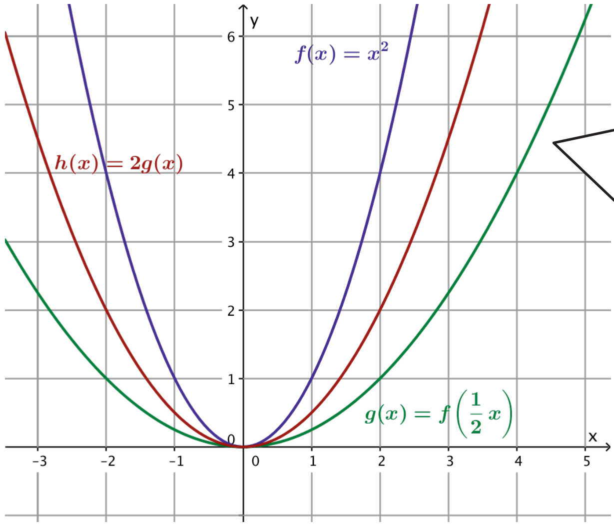
The equation for the parabola is

$$y = \frac{1}{8\sqrt{2}}\left(x - 4\sqrt{2}\right)^2 + 2\sqrt{2}.$$



Lesson 35: Are All Parabolas Similar?

1. Let $f(x) = x^2$. The graph of f is shown below.
   a. On the same axes, graph the function g, where $g(x) = f\left(\frac{1}{2}x\right)$. Then, graph the function h, where $h(x) = 2g(x)$.



I notice that applying a horizontal scaling with factor 2 and then applying a vertical scaling with factor 2 to the resulting image produces a graph that is a dilation of the graph of f with scale factor 2.

   b. Based on your work, make a conjecture about the resulting function when the original function is transformed with a horizontal scaling and then a vertical scaling by the same factor, k.

   In this example, the resulting function is a dilation with scale factor k.

2. Let $f(x) = 2x^2$.
   a. What are the focus and directrix of the parabola that is the graph of the function $f(x) = 2x^2$?

I notice that this equation is in the form $y = \frac{1}{2p}x^2$, which produces a parabola with vertex at the origin that opens upward.

Since $\frac{1}{2p} = 2$, we know that the distance between the focus and the directrix is $p = \frac{1}{4}$. The point $(0, 0)$ is both the vertex of the parabola and the midpoint of the segment connecting the focus and the directrix. Since the distance between the focus and vertex is $\frac{1}{2}p$, this distance is $\frac{1}{8}$, which is the same as the distance between the vertex and directrix. Therefore, the focus has coordinates $\left(0, \frac{1}{8}\right)$ and the directrix has equation $y = -\frac{1}{8}$.

b. Describe the sequence of transformations that would take the graph of f to each parabola described below. Then write an analytic equation for each parabola described.

i. Focus: $\left(0, -\frac{1}{8}\right)$, directrix: $y = \frac{1}{8}$

This parabola is a reflection of the graph of f across the x-axis.

$$y = -2x^2$$

I notice the distance between the focus and directrix is the same as in the graph of f and the vertex is $(0, 0)$. Since the directrix is above the focus, this transformation is a reflection across the x-axis.

ii. Focus: $(0, -\frac{3}{8})$, directrix: $y = -\frac{5}{8}$

This parabola is a vertical translation of the graph of f down $\frac{1}{2}$ unit.

$$y = 2x^2 - \frac{1}{2}$$

iii. Focus: $(1,0)$, directrix: $x = -1$

This parabola is a vertical scaling of the graph of f by a factor of $\frac{1}{8}$ and a clockwise rotation of the resulting image by $90°$ about the origin.

$$x = \frac{1}{4}y^2$$

I notice the distance between the focus and directrix is 8 times the corresponding distance on the graph of f. I also notice the directrix is vertical and to the left of the focus, which means that the graph has been rotated $90°$ clockwise.

c. Which of the graphs represent parabolas that are similar? Which represent parabolas that are congruent?

All the graphs represent similar parabolas because all parabolas are similar. The graphs of the parabolas described in parts (i) and (ii) are congruent to the original parabola because the value of p is the same.

3. Write the equation of a parabola congruent to $y = 2x^2$ that contains the point $(1, -4)$. Describe the transformations that would take this parabola to your new parabola.

$\boldsymbol{y = 2(x-1)^2 - 4}$

The graph of the equation $y = 2x^2$ has its vertex at the origin. The graph of the equation $y = 2(x-1)^2 - 4$ is a translation 1 unit to the right and 4 units down, so its vertex is $(1, -4)$.

Since I performed rigid transformations on the graph of $y = 2x^2$, the transformed graph is congruent to it.

Note: There are many possible correct answers.

Lesson 36: Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions?

1. Solve the following systems of equations, or show that no real solution exists. Graphically confirm your answers.

   a. $3x - 2y = 10$
   $6x + 4y = 16$

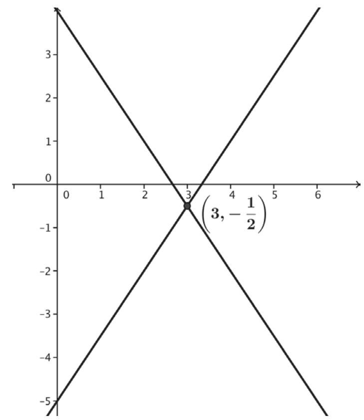
   I can solve the system using elimination like we did in Lessons 30–31.

   $$-2(3x - 2y) = -2(10)$$
   $$6x + 4y = 16$$

   $$\begin{array}{r} -6x + 4y = -20 \\ \underline{+\ 6x + 4y = 16} \\ 8y = -4 \end{array}$$

   So, $y = -\frac{1}{2}$. Then $3x - 2\left(-\frac{1}{2}\right) = 10$ so $3x + 1 = 10$ and thus $x = 3$.

   The solution is $\left(3, -\frac{1}{2}\right)$.



   b. $3x^2 + y = 1$
   $x + y = 2$

   The graph of the first equation is a parabola with vertex at $(0, 1)$ that opens down. The graph of the second equation is a line with slope -1 and y-intercept 2.

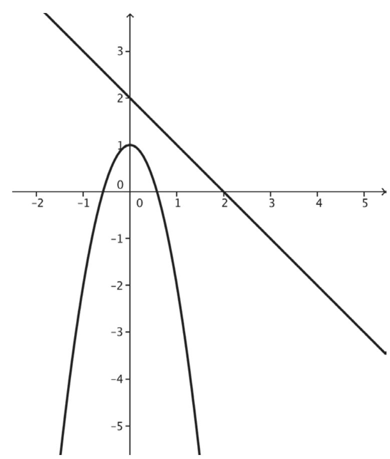
   $y = -x + 2$

   Substituting $-x + 2$ for y gives $3x^2 + (-x + 2) = 1$, so the discriminant is

   $$b^2 - 4ac = (-1)^2 - 4(3)(1)$$
   $$= -11.$$

   Since the discriminant is negative, there are no real solutions to the equation $3x^2 - x + 1 = 0$.

   There are no real solutions to the system.



   I know that given $ax^2 + bx + c = 0$, if $b^2 - 4ac < 0$, there are no real solutions to the equation. Since x represents the x-coordinate of the point of intersection of the graphs, there are no points of intersection.

2. Find the value of k so that the graph of the following system of equations has no solution.

$$2x - y + 8 = 0$$
$$kx + 3y - 9 = 0$$

If $2x - y + 8 = 0$, then $y = 2x - 8$.

If $kx + 3y - 9 = 0$, then $y = -\frac{k}{3}x + 3$.

I know when a linear equation is written in the form $y = ax + b$, a represents the slope of the line.

Since there are no solutions to the system, the lines are parallel and the slopes of the lines must be equal. Therefore, $2 = -\frac{k}{3}$, and it follows that $k = -6$.

3. Consider the equation $x^2 - 6x + 7 = 0$.

a. Offer a geometric explanation to why the equation has two real solutions.

$$\boldsymbol{x^2 - 6x + 7 = 0}$$
$$\boldsymbol{(x^2 - 6x + 9) + 7 - 9 = 0}$$
$$\boldsymbol{(x - 3)^2 - 2 = 0}$$

I can complete the square to help me identify key features of the graph of $y = x^2 - 6x + 7$.

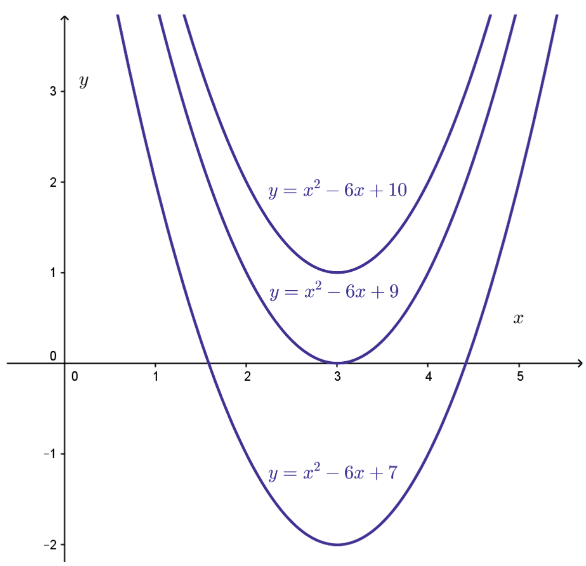
The graph of this equation is a parabola that opens upward with vertex $(3, -2)$. Since the graph opens up and has a vertex below the x-axis, the graph will intersect the x-axis in two locations, which means the equation has two real solutions.

b. Describe a geometric transformation to the graph of $y = x^2 - 6x + 7$ so the graph intersects the x-axis in one location.

A translation upward by 2 units will result in a graph that opens up with a vertex at $(3, 0)$. Since the vertex is on the x-axis and the parabola opens up, it will intersect the x-axis in only one location.

c. Describe a geometric transformation to the graph of $y = x^2 - 6x + 7$ so the graph does not intersect the x-axis.

A translation upward by 3 units will result in a parabola that opens up with a vertex at $(3, 1)$. Since the vertex is above the x-axis and the parabola opens up, the graph will not intersect the x-axis.



d. Find the equation of a line that would intersect the graph of $y = x^2 - 6x + 7$ in one location.

The graph of $y = -2$ is a horizontal line tangent to the vertex of the graph of $y = x^2 - 6x + 7$.

Note: There are multiple correct solutions to parts (b)–(d).

Lesson 37: A Surprising Boost from Geometry

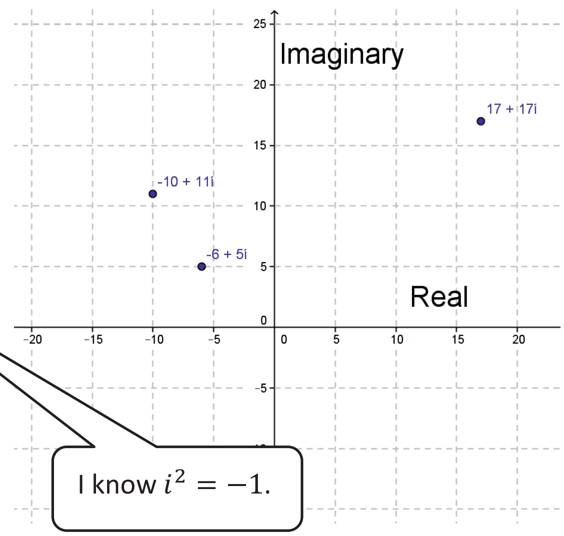
1. Express the quantities below in $a + bi$ form, and graph the corresponding points on the complex plane. Label each point appropriately.

   a. $(12 + 2i) - 3(6 - i)$

   $\mathbf{12 + 2i - 18 + 3i = -6 + 5i}$

   b. $(4 + i)(5 + 3i)$

   $$\begin{aligned} \mathbf{(4 + i)(5 + 3i)} &= \mathbf{4(5) + 4(3i) + i(5) + i(3i)} \\ &= \mathbf{20 + 12i + 5i + 3i^2} \\ &= \mathbf{20 + 12i + 5i - 3} \\ &= \mathbf{17 + 17i} \end{aligned}$$



I know $i^2 = -1$.

   c. $i(4 - i)(2 + 3i)$

   $$\begin{aligned} \mathbf{i(8 + 12i - 2i - 3i^2)} &= \mathbf{i(8 + 10i - 3(-1))} \\ &= \mathbf{i(11 + 10i)} \\ &= \mathbf{11i + 10i^2} \\ &= \mathbf{-10 + 11i} \end{aligned}$$

2. Find the real value of x and y in each of the following equations using the fact that if $a + bi = c + di$, then $a = c$ and $b = d$.

$$2(4 + x) - 5(3y + 2)i = 4x - (3 + y)i$$

Since $a = c$, it follows that $2(4 + x) = 4x$. Then $8 + 2x = 4x$, so $2x = 8$, and $x = 4$.

Since $b = d$, it follows that $-5(3y + 2) = -(3 + y)$. Then $-15y - 10 = -3 - y$, so $14y = -7$, and $y = -\frac{1}{2}$.

3. Express each of the following complex numbers in $a + bi$ form.

a. i^{23}

$$\boldsymbol{i^{23} = i^{20} \cdot i^3 = \left(i^4\right)^5 \cdot i^3 = 1 \cdot i^3 = i^2 \cdot i = -i}$$

I know
$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1.$

b. $(1 + i)^5$

$$\begin{aligned}
\boldsymbol{(1+i)^5} &\boldsymbol{= (1+i)^4(1+i)} \\
&\boldsymbol{= \left((1+i)^2\right)^2 \cdot (1+i)} \\
&\boldsymbol{= \left(1 + 2i + i^2\right)^2 (1+i)} \\
&\boldsymbol{= (1 + 2i - 1)^2 (1+i)} \\
&\boldsymbol{= (2i)^2(1+i)} \\
&\boldsymbol{= 4i^2(1+i)} \\
&\boldsymbol{= -4(1+i)} \\
&\boldsymbol{= -4 - 4i}
\end{aligned}$$

c. $\sqrt{-4} \cdot \sqrt{-25}$

$$\boldsymbol{\sqrt{-4} \cdot \sqrt{-25} = 2i \cdot 5i = 10i^2 = -10}$$

Lesson 38: Complex Numbers as Solutions to Equations

1. Solve the equation $5x^2 + 3x + 1 = 0$.

I can use the quadratic formula to solve the equation.

This is a quadratic equation with $a = 5$, $b = 3$, and $c = 1$.

$$x = \frac{-(3) \pm \sqrt{3^2 - 4(5)(1)}}{2(5)} = \frac{-3 \pm \sqrt{-11}}{10} = \frac{-3 \pm i\sqrt{11}}{10}$$

Thus, the solutions are $-\frac{3}{10} + \frac{i\sqrt{11}}{10}$ and $-\frac{3}{10} - \frac{i\sqrt{11}}{10}$.

2. Suppose we have a quadratic equation $ax^2 + bx + c = 0$ so that $a \neq 0, b = 0$, and $c < 0$. Does this equation have one solution or two distinct solutions? Are the solutions real or complex? Explain how you know.

Because $b = 0$, $ax^2 + c = 0$, so $ax^2 = -c$.

Then since $c < 0$, we know $-c > 0$, and it must be that $ax^2 > 0$.

If $a > 0$, then $x^2 > 0$, and the equation has two distinct real solutions.

I can use the multiplication property of inequality to isolate x^2.

Otherwise, if $a < 0$, then $x^2 < 0$, and there are two distinct complex solutions.

3. Write a quadratic equation in standard form such that $2i$ and $-2i$ are its solutions.

$$(x - 2i)(x + 2i) = 0$$
$$x^2 + 2ix - 2ix - 4i^2 = 0$$
$$x^2 + 4 = 0$$

I know if a is a solution to the polynomial equation $p(x) = 0$, then $(x - a)$ is a factor of p.

4. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has two real solutions if a and c are opposites?

Since a and c are opposites, $c = -a$. Then

$$b^2 - 4ac = b^2 - 4(a)(-a) = b^2 + 4a^2.$$

I know that the equation $ax^2 + bx + c = 0$ has two distinct real solutions when $b^2 - 4ac > 0$.

Since b^2 is nonnegative and $4a^2$ is positive, $b^2 + 4a^2 > 0$.

Therefore, there are always two real solutions to the equation $ax^2 + bx + c = 0$ when $a = -c$.

5. Let k be a real number, and consider the quadratic equation $kx^2 + (k-2)x + 4 = 0$.

   a. Show that the discriminant of $kx^2 + (k-2)x + 4 = 0$ defines a quadratic function of k.

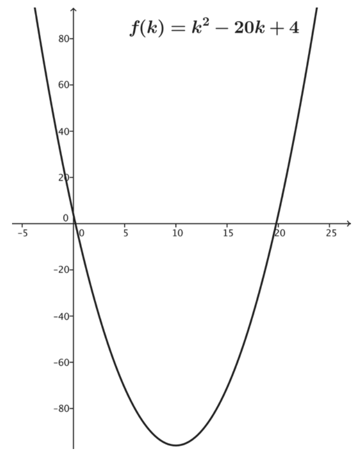
   Here, $\boldsymbol{a = k}$, $\boldsymbol{b = k - 2}$, ***and*** $\boldsymbol{c = 4}$.

   $$\begin{aligned} b^2 - 4ac &= (k-2)^2 - 4 \cdot (k) \cdot 4 \\ &= k^2 - 4k + 4 - 16k \\ &= k^2 - 20k + 4 \end{aligned}$$

   I know the discriminant of a quadratic equation written in the form $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

   With k unknown, we can write $\boldsymbol{f(k) = k^2 - 20k + 4}$, ***which is a quadratic function of k.***

   b. Find the zeros of the function in part (a), and make a sketch of its graph.

   If $\boldsymbol{f(k) = 0}$, ***then*** $\boldsymbol{0 = k^2 - 20k + 4}$.

   $$\begin{aligned} k &= \frac{-(-20) \pm \sqrt{(-20)^2 - 4(1)(4)}}{2} \\ &= \frac{20 \pm \sqrt{384}}{2} \\ &= 10 \pm 4\sqrt{6} \end{aligned}$$

   The zeros of the function are $\boldsymbol{10 + 4\sqrt{6}}$ ***and*** $\boldsymbol{10 - 4\sqrt{6}}$.

   c. For what value of k are there two complex solutions to the given quadratic equation?

   There are two complex solutions when $\boldsymbol{f(k) < 0}$.

   This occurs for all real numbers k such that $\boldsymbol{10 - 4\sqrt{6} < k < 10 + 4\sqrt{6}}$.

   I know the discriminant is less than zero where the graph of f is below the x-axis, which is between the values of the zeros.

Lesson 39: Factoring Extended to the Complex Realm

1. Rewrite each expression in standard form.

a. $(x-4i)(x+4i)$

$\boldsymbol{(x-4i)(x+4i)=x^2+4^2=x^2+16}$

I recognize this pattern as the factoring for the sum of squares: $(a+bi)(a-bi)=a^2+b^2$.

b. $(x+2-i)(x-2+i)$

$$\begin{aligned}
&\boldsymbol{(x+2-i)(x-2+i)}\\
&\quad=\boldsymbol{\big(x+(2-i)\big)\big(x-(2-i)\big)}\\
&\quad=\boldsymbol{x^2-(2-i)^2}\\
&\quad=\boldsymbol{x^2-\left(4-2i+i^2\right)}\\
&\quad=\boldsymbol{x^2-(3-2i)}\\
&\quad=\boldsymbol{x^2-3+2i}
\end{aligned}$$

I recognize this pattern as the factoring for the difference of two squares:

$$(a+b)(a-b)=a^2-b^2.$$

I can see that $a=x$ and $b=(2-i)$.

3. Write a polynomial equation of degree 4 in standard form that has the solutions $2i, -2i, 3, -2$.

I know if a is a solution to $p(x)=0$, then $(x-a)$ is a factor, so the factors must be $(x-2i)$, $(x+2i)$, $(x-3)$, and $(x+2)$.

$$\begin{aligned}
\boldsymbol{(x-2i)(x+2i)(x-3)(x+2)}&\boldsymbol{=0}\\
\boldsymbol{\left(x^2+4\right)\left(x^2-x-6\right)}&\boldsymbol{=0}\\
\boldsymbol{x^2\left(x^2-x-6\right)+4\left(x^2-x-6\right)}&\boldsymbol{=0}\\
\boldsymbol{x^4-x^3-6x^2+4x^2-4x-24}&\boldsymbol{=0}\\
\boldsymbol{x^4-x^3-2x^2-4x-24}&\boldsymbol{=0}
\end{aligned}$$

4. Find the solutions to $x^4-8x^2-9=0$ and the x-intercepts of the graph of $y=x^4-8x^2-9$.

$$\begin{aligned}
\boldsymbol{\left(x^2-9\right)\left(x^2+1\right)}&\boldsymbol{=0}\\
\boldsymbol{(x+3)(x-3)(x+i)(x-i)}&\boldsymbol{=0}
\end{aligned}$$

I can also let $u=x^2$ and factor the equation u^2-8u-9.

The solutions are -3, 3, $-i$, and i.

The x-intercepts are -3 and 3.

I know only real solutions to the equation are x-intercepts of the graph.

5. Explain how you know that the graph of $y=x^4+13x^2+36$ has no x-intercepts.

If $x^4+13x^2+36=0$, then $\left(x^2+9\right)\left(x^2+4\right)=0$, so $(x+3i)(x-3i)(x+2i)(x-2i)=0$.

The solutions to the quadratic equation are $-3i$, $3i$, $-2i$, and $2i$. Since none of the solutions are real, the graph has no x-intercepts.

Lesson 40: Obstacles Resolved—A Surprising Result

1. Write each quadratic function below as a product of of linear factors.

   a. $f(x) = 9x^2 + 16$

   $\boldsymbol{f(x) = (3x + 4i)(3x - 4i)}$

   I can factor this expression using the sum of squares identity, which we discovered in Lesson 39.

   b. $f(x) = x^2 - 6x + 12$

   $\boldsymbol{f(x) = x^2 - 6x + 12 = x^2 - 6x + 9 + 3}$

   $\boldsymbol{f(x) = (x^2 - 6x + 9) - (-3) = (x - 3)^2 - (i\sqrt{3})^2}$

   $\boldsymbol{f(x) = \left((x - 3) - i\sqrt{3}\right)\left((x - 3) + i\sqrt{3}\right)}$

   If I complete the square, I can rewrite this quadratic expression as the difference of two squares.

   Since $b^2 = -3$, I know $b = \pm\sqrt{-3} = \pm i\sqrt{3}$.

   c. $f(x) = x^3 + 10x$

   $\boldsymbol{f(x) = x\left(x^2 + 10\right) = x(x + i\sqrt{10})(x - i\sqrt{10})}$

2. For each cubic function below, one of the zeros is given. Express each function as a product of linear factors.

   a. $f(x) = 2x^3 + 3x^2 - 8x + 3$; $f(1) = 0$

   The factor theorem tells me that if $f(1) = 0$, then $(x - 1)$ is a factor of f.

   $$\boldsymbol{f(x) = (x - 1)\left(2x^2 + 5x - 3\right)}$$
   $$\boldsymbol{= (x - 1)(2x - 1)(x + 3)}$$

   I can find the quotient of f divided by $(x - 1)$ using the long division algorithm from Lesson 5 or the reverse tabular method from Lesson 4.

   b. $f(x) = x^3 + x^2 + x + 1$; $f(-1) = 0$

   $\boldsymbol{f(x) = x^2(x + 1) + 1(x + 1) = \left(x^2 + 1\right)(x + 1) = (x - i)(x + i)(x + 1)}$

   I can factor this polynomial expression by grouping.

3. Consider the polynomial function $f(x) = x^6 - 65x^3 + 64$.

   a. How many linear factors does $x^6 - 65x^3 + 64$ have? Explain.

   There are 6 linear factors because the fundamental theorem of algebra states that a polynomial with degree n can be written as a product of n linear factors.

b. Find the zeros of f.

I can factor each of these expressions using the difference of cubes pattern: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$$\begin{aligned} \boldsymbol{f(x)} &= \boldsymbol{x^6 - 65x^3 + 64} \\ &= \boldsymbol{(x^3-64)(x^3-1)} \\ &= \boldsymbol{(x-4)(x^2+4x+16)(x-1)(x^2+x+1)} \end{aligned}$$

The zero product property tells me that I can set each factor equal to 0 to find the zeros of the polynomial.

To find the zeros of $\boldsymbol{x^2+4x+4}$:

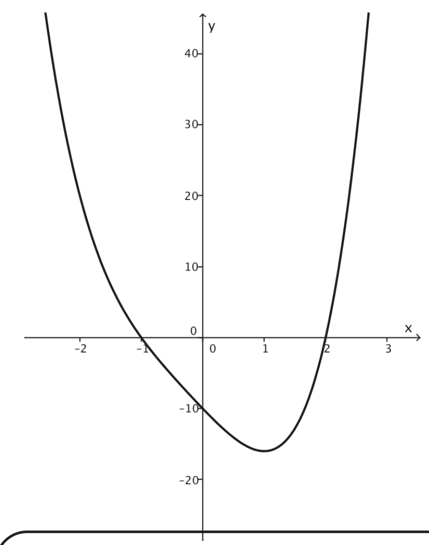
$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-64}}{2} = \frac{-4 \pm \sqrt{-48}}{2} = \frac{-4 \pm \sqrt{16}\sqrt{-3}}{2} = -2 \pm 2i\sqrt{3}}$$

To find the zeros of $\boldsymbol{x^2+x+1}$:

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.}$$

The zeros of f are $\boldsymbol{4, -2+2i\sqrt{3}, -2-2i\sqrt{3}, 1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i.}$

4. Consider the polynomial function $P(x) = x^4 + x^3 + x^2 - 9x - 10$.



a. Use the graph to find the real zeros of P.

The real zeros are $\boldsymbol{-1}$ ***and*** $\boldsymbol{2}$.

b. Confirm that the zeros are correct by evaluating the function P at those values.

$$\boldsymbol{P(-1) = (-1)^4 + (-1)^3 + (-1)^2 - 9(-1) - 10 = 0}$$
$$\boldsymbol{P(2) = (2)^4 + (2)^3 + (2)^2 - 9(2) - 10 = 0}$$

c. Express P in terms of linear factors.

$$\begin{aligned} \boldsymbol{P(x)} &= \boldsymbol{(x+1)(x-2)(x^2+2x+5)} \\ &= \boldsymbol{(x+1)(x-2)(x^2+2x+1+4)} \\ &= \boldsymbol{(x+1)(x-2)\big((x+1)^2-(2i)^2\big)} \\ &= \boldsymbol{(x+1)(x-2)\big((x-1)+2i\big)\big((x-1)-2i\big)} \end{aligned}$$

I can use the reverse tabular method or the long division algorithm to factor $(x+1)(x-2)$ from P.